Nonexistence of D(4)-quintuples

Bliznac Trebješanin, Marija; Filipin, Alan

Journal of Number Theory 194 (2019) 170–217

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Nonexistence of $D(4)$-quintuples

Marija Bliznac Trebješanin [a,*], Alan Filipin [b]

[a] University of Split, Faculty of Science, Ruđera Boškovića 33, 21000 Split, Croatia
[b] University of Zagreb, Faculty of Civil Engineering, Fra Andrije Kačića-Miošića 26, 10000 Zagreb, Croatia

ARTICLE INFO

Article history:
Received 22 January 2018
Received in revised form 4 July 2018
Accepted 30 July 2018
Available online 22 August 2018
Communicated by L. Smajlovic

MSC:
11D09
11D45
11J86

Keywords:
Diophantine m-tuples
Pell equations
Reduction method
Linear form in logarithms

ABSTRACT

In this paper we prove a conjecture that a $D(4)$-quintuple does not exist using both classical and new methods. Also, we give a new version of the Rickert's theorem that can be applied on some $D(4)$-quadruples.

1. Introduction

Definition 1. Let $n \neq 0$ be an integer. We call the set of m distinct positive integers a $D(n)$-m-tuple, if the product of any two of its distinct elements increased by n is a perfect square.

* Corresponding author.
E-mail addresses: marbli@pmfst.hr (M. Bliznac Trebješanin), filipin@grad.hr (A. Filipin).

https://doi.org/10.1016/j.jnt.2018.07.013

One of the most interesting and most studied questions is how large those sets can be. In this paper, we will consider $D(4)$-quintuples $\{a, b, c, d, e\}$ and without loss of generality we will assume that $a < b < c < d < e$. It is conjectured in [10] that all $D(4)$-quadruples, such that $a < b < c < d$, are regular, i.e.

$$d = d_+ = a + b + c + \frac{1}{2}(abc + \sqrt{(ab+4)(ac+4)(bc+4)}),$$

which implies that there does not exist a $D(4)$-quintuple.

The second author in [13] has proven that an irregular $D(4)$-quadruple cannot be extended to a quintuple with a larger element and in [14] that there are at most 4 ways to extend a $D(4)$-quadruple to a quintuple with a larger element. The best published upper bound on the number of $D(4)$-quintuples is $6.8587 \cdot 10^{29}$ found by the authors in [2].

The case $n = 1$ is the most famous and mostly studied. Dujella proved in [7] that a $D(1)$-sextuple does not exist and that there are at most finitely many quintuples. Over the years many authors improved the upper bound for the number of $D(1)$-quintuples and finally, very recently, He, Togbé and Ziegler in [16] presented the proof of the nonexistence of $D(1)$-quintuples. To see all details of the history of the problem with all references one can visit the webpage [6].

Our approach was to use the methods and approach from [16] and apply them to $D(4)$-quintuples, but modifications were necessary since not all previously proven results are comparable in the cases $n = 1$ and $n = 4$. One of the main differences is that the result from [4, Theorem A.], where the authors proved that $b > 3a$ holds for a $D(1)$-quintuple, cannot be proven for the $D(4)$ case using the analogous methods. But, in the $D(4)$ case we have $b \geq a + 57\sqrt{a}$, proven by the second author in [15], which can be used with some modifications to prove similar auxiliary results as in [16]. Throughout the paper we will give a proof only for statements which differ from the $D(1)$ case, where the modification of the proof or some new idea was necessary, or some additional explanation is needed because not all of the proofs from [16] have been explained in the way that it is clear how to apply the same method in the $D(4)$ case.

One of the sections of the paper will be dedicated to using methods from [3] to get an improved version of Rickert's theorem for $D(4)$-quadruples and use it to get the bounds on elements of a $D(4)$-quintuple in the last section of the paper which was necessary to prove our result.

The last two sections will be dedicated to proving the main result of our paper. Our main result is the following theorem.

Theorem 1. *There does not exist a $D(4)$-quintuple.*

Let us mention that a stronger version of conjecture, i.e. that all quadruples are regular, still remains open.

2. Known results about elements of a $D(4)$-m-tuple

For a $D(4)$-triple $\{a,b,c\}$, $a<b<c$, we define

$$d_{\pm}=d_{\pm}(a,b,c)=a+b+c+\frac{1}{2}(abc\pm\sqrt{(ab+4)(ac+4)(bc+4)}),$$

and it is easy to check that $\{a,b,c,d_+\}$ is a $D(4)$-quadruple, which we will call a regular quadruple, and if $d_-\neq 0$ then $\{a,b,c,d_-\}$ is also a regular $D(4)$-quadruple with $d_-<c$. Also we will use standard notation $r=\sqrt{ab+4}$, $s=\sqrt{ac+4}$ and $t=\sqrt{bc+4}$.

Lemma 1. *Let $\{a,b,c\}$ be a $D(4)$-triple and $a<b<c$. Then $c=a+b+2r$ or $c>\max\{ab,4b\}$.*

Proof. This follows from [12, Lemma 3] and [8, Lemma 1]. □

The next lemma can be proven similarly as [16, Lemma 2].

Lemma 2. *Let $\{a,b,c\}$ be a $D(4)$-triple and $a<b<c$. Then $abc+c<d_+<abc+4c$.*

Results from the next two lemmas will be used in the rest of the paper very often, so sometimes we will not reference them.

Lemma 3. *[2, Lemmas 2.2 and 2.3] Let $\{a,b,c,d,e\}$ be a $D(4)$-quintuple such that $a<b<c<d<e$. Then $b>10^5$. Also, if $c\neq a+b+2r$, then $b>4a$.*

Lemma 4. *[15, Corollary 1.2] If $\{a,b,c,d,e\}$ is a $D(4)$-quintuple such that $a<b<c<d<e$, then $b\geq a+57\sqrt{a}$.*

From [13] we also have that an element d in a $D(4)$-quintuple $\{a,b,c,d,e\}$ is uniquely determined by the triple $\{a,b,c\}$.

Lemma 5. *If $\{a,b,c,d,e\}$ is a $D(4)$-quintuple such that $a<b<c<d<e$, then $d=d_+$.*

3. New version of Rickert's theorem

In this section we will prove a new version of Rickert's theorem similar to the one in [3], which is essential to find some upper bounds on the elements of $D(4)$-quintuple when $c>a+b+2r$. Unfortunately, in the $D(4)$ case we could not get all results analogously as in [3] for a $D(1)$-quintuple, but still, these results will be essential for proving our main result.

All the results in this section and its proofs are analogous to the ones from [3] so we will give them without a proof.

Theorem 2. *Put $E' = \max\{4(F-E), 4E\}$ and $g = \gcd(E,F)$ and let E, F be integers with $0 < E/g \leq F/g - 4$, $F/g \geq 5$ and N a multiple of EF. Assume that $N \geq 59.488E'F^2(F-E)^2g^{-4}$. Then the numbers*

$$\theta_1 = \sqrt{1+\frac{4F}{N}} \quad \text{and} \quad \theta_2 = \sqrt{1+\frac{4E}{N}}$$

satisfy

$$\max\left\{\left|\theta_1 - \frac{p_1}{q}\right|, \left|\theta_2 - \frac{p_2}{q}\right|\right\} > \left(\frac{3.53081 \cdot 10^{27}E'FN}{Eg^2}\right)^{-1} q^{-\lambda}$$

for all integers p_1, p_2, $q > 0$, where

$$\lambda = 1 + \frac{\log(2.500788E^{-1}E'FNg^{-2})}{\log(0.04216N^2g^2E^{-1}F^{-1}(F-E)^{-2})} < 2.$$

Let $\{A, B, C\}$ be a $D(4)$-triple which can be extended to a quadruple with an element D. Then there exist positive integers x, y, z such that

$$AD + 4 = x^2, \quad BD + 4 = y^2, \quad CD + 4 = z^2.$$

By expressing D from these equations we get the following system of generalized Pell equations

$$Cx^2 - Az^2 = 4(C-A),$$
$$Cy^2 - Bz^2 = 4(C-B).$$

Solutions of each of these equations can be expressed with a binary recurrent sequences as described in details in [11]. We will denote them $z = v_m = w_n$, where m and n are some positive integers and we will denote the initial values of these sequences with $z_0 = v_0$ and $z_1 = w_0$. If this quadruple is contained in a $D(4)$-quintuple, then from [14] we know that m and n are even and $z_0 = z_1 = \pm 2$, so we will consider only that case.

Lemma 6. *Suppose that there exist positive integers m and n such that $z = v_{2m} = w_{2n}$, and $|z_1| = 2$, and that $C \geq B^2 \geq 25$. Then $\log z > n \log BC$.*

Proof. This lemma can be proven similarly as [11, Lemma 8] or [3, Lemma 3.1]. □

And finally we use Theorem 2 to get a new version of the Rickert's theorem.

Lemma 7. *Let A, B and C be integers, $A' = \max\{4(B-A), 4A\}$ and $g = \gcd(A,B)$. Suppose that there exist integers $m \geq 3$ and $n \geq 2$ such that $z = v_{2m} = w_{2n}$, $z_0 = z_1$,*

$|z_1| = 2$ *and that* $0 < A/g \leq B/g - 4$, $B/g \geq 5$ *and* $C \geq 59.488A'B(B-A)^2g^{-4}A^{-1}$. *Then*

$$n < \frac{4\log(8.40335 \cdot 10^{13}(A')^{\frac{1}{2}}A^{\frac{1}{2}}B^2Cg^{-1})\log(0.20533A^{\frac{1}{2}}B^{\frac{1}{2}}C(B-A)^{-1}g)}{\log(BC)\log(0.016858A(A')^{-1}B^{-1}(B-A)^{-2}Cg^4)}.$$

Proof. It is easy to see if we set $E = A$, $F = B$ and $N = ABC$ that we can apply Theorem 2 and Lemma 6 to get the upper bound on n from the statement of lemma. □

Now we will use these results to prove an upper bound on the element c in a $D(4)$-quintuple in the terms of smaller elements a and b.

Proposition 1. *Let* $\{a, b, c, d, e\}$ *be a* $D(4)$*-quintuple such that* $a < b < c < d < e$. *Then*

$$c < \frac{237.952b^3}{a}.$$

Proof. If $c = a + b + 2r$, then $c < 4b < 237.952b^3a^{-1}$.

Let us now assume that $c \neq a + b + 2r$ and that $d \geq 237.952b^4$. From Lemmas 1 and 3 we know that $b > 10^5$, $c > \max\{ab, 4b\}$ and $b > 4a$. Then

$$\frac{59.488A'B(B-A)^2}{Ag^4} < 237.952(b-a)^3 \cdot \frac{b}{a} < 237.952b^4,$$

which implies that we can use Lemma 7 for $A = a$, $B = b$ and $C = d$. Now we observe

$$8.40335 \cdot 10^{13}(A')^{\frac{1}{2}}A^{\frac{1}{2}}B^2Cg^{-1} < 8.40335 \cdot 10^{13}b^3d,$$
$$0.20533A^{\frac{1}{2}}B^{\frac{1}{2}}C(B-A)^{-1}g < 0.03423bd,$$
$$0.016858A(A')^{-1}B^{-1}(B-A)^{-2}Cg^4 > 0.0042145b^{-4}d,$$

and get

$$n < \frac{4\log(8.40335 \cdot 10^{13}b^3d)\log(0.03423bd)}{\log(bd)\log(0.0042145b^{-4}d)}.$$

It can be shown that the right hand side is decreasing in d and since $d \geq 237.952b^4$, we can now observe

$$n < \frac{4\log(1.9996 \cdot 10^{16}b^7)\log(8.14272b^5)}{\log(237.952b^5)\log(1.002848)}.$$

From the proof of [2, Lemma 3.2.] we know that in a $D(4)$-quadruple it holds $m \geq 0.618034\sqrt{d/b}$, so

$$n > 0.309017\sqrt{\frac{d}{b}} > 0.309017\sqrt{237.952}b^{3/2} > 4.7668b^{3/2}.$$

By combining the inequalities, we get $b < 803$, which cannot be true. So we have $d < 237.952b^4$ which implies $abc < 237.952b^4$, i.e.

$$c < \frac{237.952b^3}{a}. \quad \square$$

4. An operator on Diophantine triples

An operator on triples, defined for the first time by He, Togbé and Ziegler in [16], has been shown to be one of the crucial steps in proving the nonexistence of $D(1)$-quintuples. The same will be true for the $D(4)$ case, so here we define it similarly and state some analogous results concerning the operator on $D(4)$-triples. However, we slightly extend their definition.

Definition 2. A $D(4)$-triple $\{a, b, c\}$, $a < b < c$, is called an Euler or a regular triple if $c = a + b + 2r$.

For a regular triple $\{a, b, c\}$ it is easy to prove that $d_+(a, b, c) = rst$ and $s = a + r$, $t = b + r$.

The following statements about regular triples will be given without proof, since they are easy to prove as in the $D(1)$ case.

Proposition 2. *The $D(4)$-triple $\{a, b, c\}$ is a regular triple if and only if $d_-(a, b, c) = 0$.*

Proposition 3. *Let $\{a, b, c\}$ be a $D(4)$-triple, such that $a < b < c$. We have*

$$a = d_-(b, c, d_+(a, b, c)), \quad b = d_-(a, c, d_+(a, b, c)), \quad c = d_-(a, b, d_+(a, b, c)).$$

Moreover, if $\{a, b, c\}$ is not a regular triple, then

$$c = d_+(a, b, d_-(a, b, c)).$$

In particular $\{a, b, d_-(a, b, c), c\}$ is a regular $D(4)$-quadruple.

Now we will define an operator on $D(4)$-triples. The idea follows from the fact that any $D(4)$-triple can be extended with a larger element to a $D(4)$-quadruple $\{a, b, c, d_+\}$. Hence, we obtain three new $D(4)$-triples, $\{a, b, d_+\}$, $\{a, c, d_+\}$ and $\{b, c, d_+\}$ which we may consider to be farther away from a regular triple than the original triple $\{a, b, c\}$. We can reverse this observation and define the following operator.

Definition 3. We define ∂ to be an operator which sends a non-regular $D(4)$-triple $\{a, b, c\}$ to a $D(4)$-triple $\{a', b', c'\}$ such that

$$\partial(\{a, b, c\}) = \{a, b, c, d_-(a, b, c)\} \setminus \{\max(a, b, c)\}.$$

If $D(4)$-triple $\{a, b, c\}$ is a regular triple, then we define that ∂ sends this triple to the same $D(4)$-triple $\{a, b, c\}$, i.e.

$$\partial(\{a, b, c\}) = \{a, b, c\}.$$

For $D \in \mathbb{N}_0$ we can define the operator ∂_{-D} on the set of $D(4)$-triples recursively as follows.

1. For any $D(4)$-triple $\{a, b, c\}$ we define

$$\partial_0(\{a, b, c\}) = \{a, b, c\}.$$

2. We recursively define

$$\partial_{-D}(\{a, b, c\}) = \partial(\partial_{-(D-1)}(\{a, b, c\})), \quad \text{for } D \geq 1.$$

Moreover, we put

$$d_{-D}(a, b, c) = d_{-}(\partial_{-(D-1)}(\{a, b, c\})).$$

In particular, $\partial = \partial_{-1}$ and $\partial_{-2}(\{a, b, c\}) = \partial(\partial_{-1}(\{a, b, c\}))$.

Remark. Observe that by using operator ∂ repeatedly, for a fixed triple $\{a, b, c\}$ we get an infinite sequence of $D(4)$-triples

$$\partial_0(\{a, b, c\}), \partial_{-1}(\{a, b, c\}), \partial_{-2}(\{a, b, c\}), \ldots, \partial_{-D}(\{a, b, c\}), \ldots.$$

In the next Proposition we will show that for each $D(4)$-triple this sequence becomes stationary after D-th element for some D, which implies that every triple can be obtained from a regular triple using extensions with d_+ element explained before. Also, we will show that the repeating element is a regular triple, and give an upper bound for the number D.

Proposition 4. *For any fixed $D(4)$-triple $\{a, b, c\}$ there exists a minimal nonnegative integer $D < \log(abc)/\log 5$ such that $d_{-(D+1)}(a, b, c) = 0$.*

Proof. For a regular triple $\{a, b, c\}$ we have that $d_{-(D+1)}(a, b, c) = 0$ for each $D \in \mathbb{N}_0$ since $\partial_{-D}\{a, b, c\} = \{a, b, c\}$, so $D = 0$. For a non-regular triple, the idea is to use the fact that $c > abd_{-1}(a, b, c)$ and $a'b'c' = abd_{-1}(a, b, c) < abc/5$ since $ab \geq 5$. We can see that by applying k times the operator ∂ we get $a'b'c' < \frac{abc}{5^k}$, so we must have $d_{-1}(a', b', c') = 0$ for some $\{a', b', c'\}$ and the result follows from Proposition 2. □

Definition 4. For a $D(4)$-triple $\{a, b, c\}$ we will say that it has a degree D and that it is generated by a regular triple $\{a', b', c'\}$ if D is minimal nonnegative integer such that $d_{-(D+1)}(a, b, c) = 0$ and $\partial_{-D}(\{a, b, c\}) = \{a', b', c'\}$. If the triple $\{a, b, c\}$ is of degree D we will write $\deg(a, b, c) = D$.

Remark. Let us now observe an example of these definitions. The $D(4)$-triple $\{1, 5, 12\}$ generates 3 triples of degree 1, $\{1, 5, 96\}$, $\{1, 12, 96\}$ and $\{5, 12, 96\}$, and 9 triples of degree 2, one of them is $\{1, 12, 1365\}$. It is not hard to see by induction that each $D(4)$-triple generates 3^k distinct triples of degree k. More precisely, we can see from the definition that all triples of degree 1 are distinct. Let us assume that all triples of degree k are distinct and let us observe two triples of degree $k + 1$, namely $\{a, b, c\}$ and $\{a', b', c'\}$. It is enough to notice that triples $\partial_{-1}(\{a, b, c\})$ and $\partial_{-1}(\{a', b', c'\})$ of degree k are either the same triple or they are distinct. In the former case, we conclude as in the case when degree is 1 that $\{a, b, c\}$ and $\{a', b', c'\}$ are distinct. In the latter case, if $\{a, b, c\}$ and $\{a', b', c'\}$ were not distinct that would imply that $\partial_{-1}(\{a, b, c\})$ and $\partial_{-1}(\{a', b', c'\})$ were equal, which would lead to a contradiction.

5. System of Pell equations

Let $\{a, b, c\}$ be a $D(4)$-triple, $a < b < c$, and r, s, t positive integers such that

$$ab + 4 = r^2, \quad ac + 4 = s^2, \quad bc + 4 = t^2.$$

Suppose that $\{a, b, c, d, e\}$ is a $D(4)$-quintuple, $a < b < c < d < e$, and as before

$$ad + 4 = x^2, \quad bd + 4 = y^2, \quad cd + 4 = z^2,$$

$x, y, z \in \mathbb{N}$. Then, there also exist integers X, Y, Z, W such that

$$ae + 4 = X^2, \quad be + 4 = Y^2, \quad ce + 4 = Z^2, \quad de + 4 = W^2.$$

From [13, Theorem 1] we have $d = d_+$, which implies

$$x = \frac{at + rs}{2}, \quad y = \frac{bs + rt}{2}, \quad z = \frac{cr + st}{2}.$$

By eliminating e from the equations above, we get a system of generalized Pell equations

$$aY^2 - bX^2 = 4(a - b), \tag{1}$$

$$aZ^2 - cX^2 = 4(a - c), \tag{2}$$

$$bZ^2 - cY^2 = 4(b - c), \tag{3}$$

$$aW^2 - dX^2 = 4(a - d), \tag{4}$$

$$bW^2 - dY^2 = 4(b-d), \tag{5}$$

$$cW^2 - dZ^2 = 4(c-d). \tag{6}$$

The next lemma, which is a part of Lemma 2 in [10], gives us a description of solutions of Pell equations (1)–(6).

Lemma 8. *If (X,Y) is a positive integer solution to a generalized Pell equation*

$$aY^2 - bX^2 = 4(a-b),$$

with $ab+4=r^2$, then it is obtained from

$$Y\sqrt{a} + X\sqrt{b} = (y_0\sqrt{a} + x_0\sqrt{b})\left(\frac{r+\sqrt{ab}}{2}\right)^n,$$

where $n \geq 0$ is an integer and (x_0, y_0) is integer solution of the equation such that

$$1 \leq x_0 \leq \sqrt{\frac{a(b-a)}{r-2}}, \quad \text{and} \quad 1 \leq |y_0| \leq \sqrt{\frac{(r-2)(b-a)}{a}}.$$

By applying this Lemma to the equations (1)–(6) we obtain

$$Y\sqrt{a} + X\sqrt{b} = Y_{h'}^{(a,b)}\sqrt{a} + X_{h'}^{(a,b)}\sqrt{b} = (Y_0\sqrt{a} + X_0\sqrt{b})\left(\frac{r+\sqrt{ab}}{2}\right)^{h'} \tag{7}$$

$$Z\sqrt{a} + X\sqrt{c} = Z_{j'}^{(a,c)}\sqrt{a} + X_{j'}^{(a,c)}\sqrt{c} = (Z_1\sqrt{a} + X_1\sqrt{c})\left(\frac{s+\sqrt{ac}}{2}\right)^{j'} \tag{8}$$

$$Z\sqrt{b} + Y\sqrt{c} = Z_{k'}^{(b,c)}\sqrt{b} + Y_{k'}^{(b,c)}\sqrt{c} = (Z_2\sqrt{b} + Y_2\sqrt{c})\left(\frac{t+\sqrt{bc}}{2}\right)^{k'} \tag{9}$$

$$W\sqrt{a} + X\sqrt{d} = W_{l'}^{(a,d)}\sqrt{a} + X_{l'}^{(a,d)}\sqrt{d} = (W_3\sqrt{a} + X_3\sqrt{d})\left(\frac{x+\sqrt{ad}}{2}\right)^{l'} \tag{10}$$

$$W\sqrt{b} + Y\sqrt{d} = W_{m'}^{(b,d)}\sqrt{b} + Y_{m'}^{(b,d)}\sqrt{d} = (W_4\sqrt{b} + Y_4\sqrt{d})\left(\frac{y+\sqrt{bd}}{2}\right)^{m'} \tag{11}$$

$$W\sqrt{c} + Z\sqrt{d} = W_{n'}^{(c,d)}\sqrt{c} + Z_{n'}^{(c,d)}\sqrt{d} = (W_5\sqrt{c} + Z_5\sqrt{d})\left(\frac{z+\sqrt{cd}}{2}\right)^{n'} \tag{12}$$

where h', j', k', l', m', n' are nonnegative integers, and Y_0, Y_2, Y_4, X_0, X_1, X_3, Z_1, Z_2, Z_5, W_3, W_4, W_5 are integers which satisfy appropriate inequalities from Lemma 8. Each sequence of solutions can be expressed as a pair of binary recurrence sequences, so for example, a sequence of solutions $(Y_{h'}^{(a,b)}, X_{h'}^{(a,b)})$ to equation (7) satisfy the following recursions:

$$Y_0^{(a,b)} = Y_0, \quad Y_1^{(a,b)} = \frac{rY_0 + bX_0}{2}, \quad Y_{h'+2}^{(a,b)} = rY_{h'+1}^{(a,b)} - Y_{h'}^{(a,b)},$$

$$X_0^{(a,b)} = X_0, \quad X_1^{(a,b)} = \frac{rX_0 + aY_0}{2}, \quad X_{h'+2}^{(a,b)} = rX_{h'+1}^{(a,b)} - X_{h'}^{(a,b)},$$

which can easily be proven by induction.

We will now state and prove some lemmas about initial values of the sequences of solutions and about its indices h', j', l', k', m', n'.

Lemma 9. *[14, Lemma 3] If* $W = W_{l'}^{(a,d)} = W_{m'}^{(b,d)} = W_{n'}^{(c,d)}$*, then we have* $l' \equiv m' \equiv n' \equiv 0 \pmod 2$*. Also,*

$$W_3 = W_4 = W_5 = 2\varepsilon = \pm 2 \quad \textit{and} \quad X_3 = Y_4 = Z_5 = 2.$$

In the next lemma we will prove a similar result for the remaining indices and initial values of sequences. The proof defers from the one in [16] so we give it in detail.

Lemma 10. *We have* $h' \equiv j' \equiv k' \equiv 0 \pmod 2$ *and*

$$X_0 = X_1 = Y_0 = Y_2 = Z_1 = Z_2 = 2.$$

Proof. Let us consider the system of the equations (1) and (5).

From Lemma 8 we have the bound on Y_0, $|Y_0| < b^{3/4}a^{-1/4}$, and since $Y_{h'}^{(a,b)}$ satisfies the recursion

$$Y_0^{(a,b)} = Y_0, \quad Y_1^{(a,b)} = \frac{rY_0 + bX_0}{2}, \quad Y_{h'+2}^{(a,b)} = rY_{h'+1}^{(a,b)} - Y_{h'}^{(a,b)},$$

we easily see that

$$Y_{h'}^{(a,b)} \equiv \begin{cases} Y_0^{(a,b)} \pmod b, & h' \text{ even}, \\ Y_1^{(a,b)} \pmod b, & h' \text{ odd}. \end{cases}$$

On the other hand, for $Y_{m'}^{(b,d)}$, from Lemma 9, we have

$$Y_0^{(b,d)} = Y_4 = 2, \quad Y_1^{(b,d)} = y + \varepsilon b, \quad Y_{m'+2}^{(b,d)} = yY_{m'+1}^{(b,d)} - Y_{m'}^{(b,d)}$$

and since we know that m' is even, we obtain $Y_{m'}^{(b,d)} \equiv 2 \pmod b$.

We consider $Y_{h'}^{(a,b)} = Y_{m'}^{(b,d)}$ and let us assume that h' is odd. Then

$$\frac{1}{2}(rY_0 + bX_0) \equiv 2 \pmod b$$

and since $bX_0 \equiv 0 \pmod{b}$, we have

$$bX_0 - \frac{1}{2}(rY_0 + bX_0) \equiv -2 \pmod{b},$$

i.e. $\frac{1}{2}(bX_0 - rY_0) \equiv -2 \pmod{b}$. Now, we observe

$$\begin{aligned}(bX_0 - rY_0)(bX_0 + rY_0) &= b^2X_0^2 - r^2Y_0^2 = b(aY_0^2 + 4(b-a)) - abY_0^2 - 4Y_0^2 \\ &= 4b(b-a) - 4Y_0^2.\end{aligned}$$

Since $|Y_0| < b^{3/4}a^{-1/4}$, we have

$$4b(b-a) - 4Y_0^2 > 4b(b-a) - 4b^{3/2}a^{-1/2} = 4b(b - a - (b/a)^{1/2}),$$

and since the right hand side is increasing in b, and from Lemma 4 we know that $b \geq a + 57\sqrt{a}$, we get

$$b - a - (b/a)^{1/2} > 57\sqrt{a} - \left(1 + \frac{57}{\sqrt{a}}\right)^{1/2} > 0.$$

So we can conclude that $bX_0 - r|Y_0| > 0$. On the other hand, we can easily see that $b^2X_0^2 - r^2Y_0^2 < 4b^2$, i.e. $\frac{1}{2}(bX_0 - r|Y_0|) < b$.

Now, let us consider the following two cases:

1.) If $Y_0 > 0$, then $\frac{1}{2}(bX_0 - r|Y_0|) = \frac{1}{2}(bX_0 - rY_0)$ so we must have $\frac{1}{2}(bX_0 - rY_0) = b-2$. Observe that

$$\begin{aligned}bX_0 + rY_0 &< \frac{4b^2}{bX_0 - rY_0} = \frac{4b^2}{2b-4} \\ &= 2b + \frac{8b}{2b-4} = 2b + 4 + \frac{16}{2b-4} < 2b + 4.1.\end{aligned}$$

Since $b > 10^5$ implies $r > 316$ and both addends on the right hand side of the inequality are positive, the only options for X_0 are $X_0 = 1$ and $X_0 = 2$. If $X_0 = 1$, by direct computation we can see that there is no Y_0 in the bounds given by Lemma 8 that satisfy equation (1). For $X_0 = 2$ we get only $Y_0 = 2$. But, then we would have $\frac{1}{2}(2b - 2r) = b - r = b - 2$, i.e. $r = 2$ which cannot be.

2.) If $Y_0 < 0$, then $\frac{1}{2}(bX_0 - r|Y_0|) = \frac{1}{2}(bX_0 + rY_0)$ so we have $\frac{1}{2}(bX_0 + rY_0) = 2$. This implies

$$bX_0 - rY_0 < 2bX_0 < 2b\sqrt{\frac{a(b-a)}{r-2}} < 2b\sqrt{b},$$

since $a < r - 2$ (otherwise we would get $b \leq a + 4$, which is in a contradiction with Lemma 4). We also have that $4b(b-a) - 4Y_0^2 > 4b^2 - 4ab - 4b\sqrt{b} = 4b(b - a - \sqrt{b})$, since $Y_0^2 < b^{3/2}$, so we can conclude

$$4 = bX_0 + rY_0 > \frac{4b(b-a-\sqrt{b})}{bX_0 - rY_0} > \frac{4b(b-a-\sqrt{b})}{2b\sqrt{b}} = \frac{2}{\sqrt{b}}(b-a-\sqrt{b}),$$

i.e. $\sqrt{b} - 1 - \frac{a}{\sqrt{b}} < 2$.

After squaring this expression and solving the quadratic inequality in b we get $b < a + \frac{3}{2}(\sqrt{4a+9}+3)$. Again, by Lemma 4 we also have $b \geq a + 57\sqrt{a}$, and from these two inequalities we would get $a < 1$, a contradiction.

Hence, h' must be even. From $Y_0 \equiv 2 \pmod{b}$ and $|Y_0| < b^{3/4}$ we conclude $Y_0 = 2$ and by direct computation from (1) we also get $X_0 = 2$.

Now, we consider the system of equations (2) and (6). The proof is very similar to the previous case, so we omit details and only emphasize that here we use $c \geq a + b + 2r$ to get a contradiction in the case that j' is odd. The same argument is used to prove that k' is even when we consider the system of equations (3) and (6). □

From the previous lemmas we see that equations (7)–(12) actually have form:

$$Y\sqrt{a} + X\sqrt{b} = (2\sqrt{a} + 2\sqrt{b})\left(\frac{r+\sqrt{ab}}{2}\right)^{2h}, \tag{13}$$

$$Z\sqrt{a} + X\sqrt{c} = (2\sqrt{a} + 2\sqrt{c})\left(\frac{s+\sqrt{ac}}{2}\right)^{2j}, \tag{14}$$

$$Z\sqrt{b} + Y\sqrt{c} = (2\sqrt{b} + 2\sqrt{c})\left(\frac{t+\sqrt{bc}}{2}\right)^{2k}, \tag{15}$$

$$W\sqrt{a} + X\sqrt{d} = (2\varepsilon\sqrt{a} + 2\sqrt{d})\left(\frac{x+\sqrt{ad}}{2}\right)^{2l}, \tag{16}$$

$$W\sqrt{b} + Y\sqrt{d} = (2\varepsilon\sqrt{b} + 2\sqrt{d})\left(\frac{y+\sqrt{bd}}{2}\right)^{2m}, \tag{17}$$

$$W\sqrt{c} + Z\sqrt{d} = (2\varepsilon\sqrt{c} + 2\sqrt{d})\left(\frac{z+\sqrt{cd}}{2}\right)^{2n}. \tag{18}$$

6. Gap principle and classical congruences

We have already observed, if there exists a nonnegative integer e such that the $D(4)$-quadruple $\{a, b, c, d\}$ can be extended to the quintuple $\{a, b, c, d, e\}$ then the equalities (13)–(18) are satisfied for some nonnegative integers h, j, k, l, m, and n. We will now state and prove which relations hold between these indices, but first we will state without proof some known relations.

Lemma 11. *[12, Lemma 5] If $Z = Z_{2j}^{(a,c)} = Z_{2k}^{(b,c)}$, then $k - 1 \leq j \leq 2k + 1$.*

Lemma 12. *[14, Lemma 4] If* $W = W_{2l}^{(a,d)} = W_{2m}^{(b,d)} = W_{2n}^{(c,d)}$*, then* $8 \le n \le m \le l \le 2n$*.*

As we can see, so far no one has considered relations between h and other indices. We will now prove which relation holds between m and h and improve the relation between m and l.

Lemma 13. *We have* $2l \le 3m$ *and* $m < l$*, for* $m \ge 2$*.*

Proof. From (10) and (11) by expressing solutions explicitly we have

$$W_l^{(a,d)} = \frac{d + \varepsilon\sqrt{ad}}{\sqrt{ad}} \left(\frac{x + \sqrt{ad}}{2} \right)^l + \frac{-d + \varepsilon\sqrt{ad}}{\sqrt{ad}} \left(\frac{x - \sqrt{ad}}{2} \right)^l,$$

and

$$W_m^{(b,d)} = \frac{d + \varepsilon\sqrt{bd}}{\sqrt{bd}} \left(\frac{y + \sqrt{bd}}{2} \right)^m + \frac{-d + \varepsilon\sqrt{bd}}{\sqrt{bd}} \left(\frac{y - \sqrt{bd}}{2} \right)^m.$$

Firstly, let us prove that $2l \le 3m$ by observing that we must have $W_{2l}^{(a,d)} = W_{2m}^{(b,d)}$.

Notice that $x - \sqrt{ad} < 1$, which implies that also $\left(\frac{x-\sqrt{ad}}{\sqrt{ad}}\right)^{2m} < 1$ and since $-d + \varepsilon\sqrt{ad} < 0$, we have

$$\frac{-d + \varepsilon\sqrt{ad}}{\sqrt{ad}} \left(\frac{x - \sqrt{ad}}{\sqrt{ad}} \right)^{2m} > \frac{-d + \varepsilon\sqrt{ad}}{\sqrt{ad}} \ge \frac{-d - \sqrt{ad}}{\sqrt{ad}}.$$

Moreover, notice that the second addend in the expressions for $W_l^{(a,d)}$ and $W_m^{(b,d)}$, respectively, is negative since $d > b > a$, i.e. $d > \sqrt{bd} > \sqrt{ad}$. Now, it is easy to see that

$$\frac{d + \varepsilon\sqrt{ad}}{\sqrt{ad}} \left(\frac{x + \sqrt{ad}}{2} \right)^{2l} - \frac{d + \sqrt{ad}}{\sqrt{ad}} < W_{2l}^{(a,d)} = W_{2m}^{(b,d)} <$$

$$< \frac{d + \varepsilon\sqrt{bd}}{\sqrt{bd}} \left(\frac{y + \sqrt{bd}}{2} \right)^{2m}.$$

On the other hand,

$$\frac{d + \sqrt{ad}}{\sqrt{ad}} = 1 + \frac{d}{\sqrt{ad}} < \left(\frac{y + \sqrt{bd}}{2} \right)^2 < \left(\frac{y + \sqrt{bd}}{2} \right)^{2m},$$

so we get the inequality

$$\frac{d+\varepsilon\sqrt{ad}}{\sqrt{ad}}\left(\frac{x+\sqrt{ad}}{2}\right)^{2l} < \left(\frac{d+\varepsilon\sqrt{bd}}{\sqrt{bd}}+1\right)\left(\frac{y+\sqrt{bd}}{2}\right)^{2m}$$

which implies that

$$\left(\frac{x+\sqrt{ad}}{2}\right)^{2l} < \sqrt{\frac{a}{b}}\cdot\frac{d+(\varepsilon+1)\sqrt{bd}}{d+\varepsilon\sqrt{ad}}\left(\frac{y+\sqrt{bd}}{2}\right)^{2m}.$$

From $\sqrt{a/b}(d+(\varepsilon+1)\sqrt{bd}) \leq \sqrt{a/b}(d+2\sqrt{bd}) = \sqrt{a/b}\cdot d+2\sqrt{ad} < d+2\sqrt{ad}$ we have

$$\left(\frac{x+\sqrt{ad}}{2}\right)^{2l} < \frac{d+2\sqrt{ad}}{d+\varepsilon\sqrt{ad}}\left(\frac{y+\sqrt{bd}}{2}\right)^{2m}.$$

Assume now the opposite, i.e. that $2l \geq 3m+1$. Then, we have

$$\left(\frac{x+\sqrt{ad}}{2}\right)^{3m+1} < \frac{d+2\sqrt{ad}}{d+\varepsilon\sqrt{ad}}\left(\frac{y+\sqrt{bd}}{2}\right)^{2m}$$

and the inequality $(x+\sqrt{ad})/2 > (d+2\sqrt{ad})/(d+\varepsilon\sqrt{ad})$ implies

$$\left(\frac{x+\sqrt{ad}}{2}\right)^{3} < \left(\frac{y+\sqrt{bd}}{2}\right)^{2}.$$

Since $x+\sqrt{ad} > 2\sqrt{ad}$ and $\sqrt{bd+4} < \sqrt{bd}+2/\sqrt{bd}$, we have $y+\sqrt{bd} < 2\sqrt{bd}+2/\sqrt{bd} < 2\sqrt{bd}\,(1+1/(bd)) < 2\sqrt{bd}\left(1+1/B_0^3\right)$ where $B_0 < b$. Now we get to observe the inequality

$$(\sqrt{ad})^3 < (\sqrt{bd})^2\left(1+\frac{1}{B_0^3}\right)^2$$

and after squaring, inserting $abc < d$ and canceling we get

$$a^4c < b\left(1+\frac{1}{B_0^3}\right)^4.$$

For $B_0 = 10^5$, we see that the inequality cannot be true for $a > 1$ or for $c > 4b$. It remains to investigate the case where $a = 1$ and $c = a+b+2r$. In this case we have

$$1+b+2\sqrt{b+4} < b\left(1+\frac{1}{B_0^3}\right)^4$$

and for $B_0 = 10^5$ we get $b > 2.5\cdot 10^{29}$ and that value can be used as a new value for B_0. After inserting this value we get an inequality which has no positive integer solution b.

So each case leads to a contradiction, which implies that our assumption was wrong, i.e. we have $2l \leq 3m$.

Now we assume $m = l$. Similarly as before, we observe that we have

$$\frac{d+\varepsilon\sqrt{bd}}{\sqrt{bd}}\left(\frac{y+\sqrt{bd}}{2}\right)^{2m} - \frac{d+\sqrt{bd}}{\sqrt{bd}} < W_{2m}^{(b,d)} = W_{2m}^{(a,d)}$$
$$< \frac{d+\varepsilon\sqrt{ad}}{\sqrt{ad}}\left(\frac{x+\sqrt{ad}}{2}\right)^{2m}$$

and since $d+\sqrt{bd} < ((y+\sqrt{bd})/2)^2$, we get

$$\frac{d+\varepsilon\sqrt{bd}-1}{\sqrt{bd}}\left(\frac{y+\sqrt{bd}}{2}\right)^{2m} < \frac{d+\varepsilon\sqrt{ad}}{\sqrt{ad}}\left(\frac{x+\sqrt{ad}}{2}\right)^{2m}$$

and after multiplying and rearranging we get

$$\left(\frac{y+\sqrt{bd}}{x+\sqrt{ad}}\right)^{2m} < \sqrt{\frac{b}{a}} \cdot \frac{d+\varepsilon\sqrt{ad}}{d+\varepsilon\sqrt{bd}-1}.$$

But, we have

$$\frac{d+\varepsilon\sqrt{ad}}{d+\varepsilon\sqrt{bd}-1} < \frac{d+\sqrt{bd}}{d-\sqrt{bd}} = 1 + \frac{2\sqrt{bd}}{d-\sqrt{bd}} = 1 + \frac{2}{\frac{d}{\sqrt{bd}}-1}$$
$$< 1 + \frac{2}{\sqrt{B_0}-1} = \frac{\sqrt{B_0}+1}{\sqrt{B_0}-1},$$

where the last inequality is true since $d/\sqrt{bd} = \sqrt{d/b} > \sqrt{abc/b} = \sqrt{ac} > \sqrt{b} > \sqrt{B_0}$. So, it must hold

$$\left(\frac{y+\sqrt{bd}}{x+\sqrt{ad}}\right)^{2m} < \sqrt{\frac{b}{a}} \cdot \frac{\sqrt{B_0}+1}{\sqrt{B_0}-1}.$$

On the other hand, it is easy to see that

$$\left(\frac{y+\sqrt{bd}}{x+\sqrt{ad}}\right)^{2} > \sqrt{\frac{b}{a}},$$

which, with the previous inequality, leads to the conclusion that

$$\left(\frac{b}{a}\right)^{(m-1)/2} < \frac{\sqrt{B_0}+1}{\sqrt{B_0}-1}.$$

From [2, Lemma 3.2] we can conclude that $l' = 2l > 0.61803d^{1/4} > 0.61803 \cdot 10^{10/4} > 195$. Also, from Lemma 11 we have $2l \leq 4m+1$ so $m > 48$. Now we observe

$$\left(\frac{b}{a}\right)^{47/2} < \frac{\sqrt{B_0}+1}{\sqrt{B_0}-1}$$

i.e.

$$(a+57\sqrt{a})^{47/2} < b^{47/2} < \frac{\sqrt{B_0}+1}{\sqrt{B_0}-1}a^{47/2}$$

and by solving this inequality in a for $B_0 = 10^5$ we obtain $a > 4.484 \cdot 10^{10}$ which can be used as a new value for B_0, since $b > a$. By iterating this process we get a contradiction, this time a contradiction with the upper bound $b < 10^{36}$ from [2]. We can now conclude $m \neq l$. □

Lemma 14. *We have $h \geq 2m$.*

Proof. Similarly as in the previous Lemma, for sequences $Y_{2h}^{(a,b)}$ and $Y_{2m}^{(b,d)}$ we have

$$Y_{2h}^{(a,b)} = \frac{b+\sqrt{ab}}{\sqrt{ab}}\left(\frac{r+\sqrt{ab}}{2}\right)^{2h} + \frac{-b+\sqrt{ab}}{\sqrt{ab}}\left(\frac{r-\sqrt{ab}}{2}\right)^{2h},$$

$$Y_{2m}^{(b,d)} = \frac{\varepsilon b+\sqrt{bd}}{\sqrt{bd}}\left(\frac{y+\sqrt{bd}}{2}\right)^{2m} + \frac{-\varepsilon b+\sqrt{bd}}{\sqrt{bd}}\left(\frac{y-\sqrt{bd}}{2}\right)^{2m}.$$

If $Y = Y_{2h}^{(a,b)} = Y_{2m}^{(b,d)}$, we have

$$\left(1-\sqrt{b/d}\right)\left(\frac{y+\sqrt{bd}}{2}\right)^{2m} < \frac{-b+\sqrt{bd}}{\sqrt{bd}}\left(\frac{y+\sqrt{bd}}{2}\right)^{2m} < Y_{2m}^{(b,d)} = Y_{2h}^{(a,b)} <$$

$$< \frac{b+\sqrt{ab}}{\sqrt{ab}}\left(\frac{r+\sqrt{ab}}{2}\right)^{2h} \leq \left(1+\sqrt{b/a}\right)\left(\frac{r+\sqrt{ab}}{2}\right)^{2h}.$$

It is easy to see that $\frac{\sqrt{d}(\sqrt{a}+\sqrt{b})}{\sqrt{a}(\sqrt{d}-\sqrt{b})} < \frac{r+\sqrt{ab}}{2}$, so we have

$$\left(\frac{y+\sqrt{bd}}{2}\right)^{2m} < \frac{1+\sqrt{b/a}}{1-\sqrt{b/d}}\left(\frac{r+\sqrt{ab}}{2}\right)^{2h} < \left(\frac{r+\sqrt{ab}}{2}\right)^{2h+1}.$$

Since

$$\frac{y+\sqrt{bd}}{2} > \sqrt{bd} > \sqrt{ab^2c} \geq \sqrt{ab^2(a+b+2r)} > r^2 > \left(\frac{r+\sqrt{ab}}{2}\right)^2,$$

we get

$$\left(\frac{r+\sqrt{ab}}{2}\right)^{4m} < \left(\frac{y+\sqrt{bd}}{2}\right)^{2m} < \left(\frac{r+\sqrt{ab}}{2}\right)^{2h+1},$$

and it is easy to conclude $4m < 2h+1$, i.e. $2m \le h$. □

Now, we will shortly prove classical congruences associated with $D(4)$-quintuples.

Lemma 15. *Let $\{a,b,c,d,e\}$ be a $D(4)$-quintuple. Then*

$$a\varepsilon l^2 + xl \equiv b\varepsilon m^2 + ym \equiv c\varepsilon n^2 + zn \pmod{d}.$$

Proof. If we observe the sequence $W_{2l}^{(a,d)}$ we see that

$$\begin{aligned} W_{2l+2} &= xW_{2l+1} - W_{2l} = x^2W_{2l} - (xW_{2l-1} - W_{2l-2}) - W_{2l-2} - W_{2l} = \\ &= x^2W_{2l} - W_{2l} - W_{2l-2} - W_{2l} = (x^2-2)W_{2l} - W_{2l-2}. \end{aligned}$$

As in [10, Lemma 3] it is easy to prove

$$W_{2l}^{(a,d)} \equiv 2\varepsilon + d(a\varepsilon l^2 + xl) \pmod{d^2},$$

and since $W = W_{2l}^{(a,d)} = W_{2m}^{(b,d)} = W_{2n}^{(c,d)}$, and analogous results hold for all sequences, for a $D(4)$-quintuple we get

$$a\varepsilon l^2 + xl \equiv b\varepsilon m^2 + ym \equiv c\varepsilon n^2 + zn \pmod{d}. \quad \square$$

Unfortunately, using these congruences and methods from [16] we could not get $m > \alpha\sqrt{d/b}$ for some coefficient α as "large" as the one proved for $D(1)$-quintuples in [16]. Our largest possible α was obtained after adjusting the method from [5, Proposition 3.1], which we have also used in [2] to get a similar coefficient for $D(4)$-quadruples. We omit the proof since it is similar to the one given in detail in [5].

Lemma 16. *Let $\{a,b,c,d,e\}$ be a $D(4)$-quintuple such that $a<b<c<d<e$, $W_{2l}^{(a,b)} = W_{2m}^{(b,d)}$ and $\frac{3}{2}m \ge l > m \ge 2$. Assume that $a \ge A_0$, $b \ge B_0$ and $d \ge D_0$, $b > \rho a$, $\rho \ge 1$. Then*

$$l > \alpha b^{-1/2} d^{1/2}$$

for every real number α that satisfy both inequalities

$$\alpha^2 + \alpha(1 + 2B_0^{-1}D_0^{-1}) \le 1, \tag{19}$$

$$\frac{20}{9}\alpha^2 + \alpha(B_0(\lambda + \rho^{-1/2}) + 2D_0^{-1}(\lambda + \rho^{1/2})) \le B_0, \tag{20}$$

where $\lambda = \sqrt{\frac{A_0+4}{\rho A_0+4}}$.

Now we use this result to get lower bounds on indices in the terms of ac.

Lemma 17. *Let $\{a, b, c, d, e\}$ be a $D(4)$-quintuple. Then $l > 0.499997\sqrt{ac}$, $j > m > 0.333331\sqrt{ac}$ and $h > 0.666662\sqrt{ac}$.*

Proof. By inserting $\rho = 1$, $A_0 = 1$, $B_0 = 10^5$ and $D_0 = 10^{10}$ in the inequalities from Lemma 16 we compute that $\alpha = 0.499997$ satisfies both inequalities (19) and (20). The statement now follows from Lemmas 12, 13 and 14 and the fact that $d > abc$. □

7. Linear forms in logarithms

In this section we use different methods to find a good upper bound on the index h and the product ac in a $D(4)$-quintuple. Even though many authors usually apply Matveev's theorem on a linear form in logarithms from [18], we will use Aleksentsev's version of the theorem from [1] as authors in [5] did and which we also applied in [2] because it will give us slightly better bounds.

For any non-zero algebraic number γ of degree D over $\mathbb{Q}$, with minimal polynomial $A\prod_{j=1}^{D}\left(X - \gamma^{(j)}\right)$ over $\mathbb{Z}$, we define its absolute logarithmic height as

$$h(\gamma) = \frac{1}{D}\left(\log A + \sum_{j=1}^{D}\log^{+}\left|\left(\gamma^{(j)}\right)\right|\right),$$

where $\log^{+}\alpha = \log\max\{1, \alpha\}$.

Theorem 3 *(Aleksentsev). Let Λ be a linear form in logarithms of n multiplicatively independent totally real algebraic numbers $\alpha_1, \dots, \alpha_n$, with rational integer coefficients $b_1, \dots, b_n$. Let $h(\alpha_j)$ denote the absolute logarithmic height of α_j for $1 \le j \le n$. Let d be the degree of the number field $\mathcal{K} = \mathbb{Q}(\alpha_1, \dots, \alpha_n)$, and let $A_j = \max\left(dh(\alpha_j), |\log\alpha_j|, 1\right)$. Finally, let*

$$E = \max\left(\max_{1\le i,j\le n}\left\{\frac{|b_i|}{A_j} + \frac{|b_j|}{A_i}\right\}, 3\right). \tag{21}$$

Then

$$\log|\Lambda| \ge -5.3n^{\frac{1-2n}{2}}(n+1)^{n+1}(n+8)^2(n+5)31.44^n d^2(\log E)A_1\cdots A_n\log(3nd).$$

Let us define the linear form in logarithms

$$\Lambda_1 := 2h\log\frac{r+\sqrt{ab}}{2} - 2j\log\frac{s+\sqrt{ac}}{2} + \log\frac{\sqrt{c}(\sqrt{a}+\sqrt{b})}{\sqrt{b}(\sqrt{a}+\sqrt{c})}.$$

Analogously as in [16, Lemma 17] we can find the bounds for Λ_1.

Lemma 18. *We have* $0 < \Lambda_1 < \left(\frac{s+\sqrt{ac}}{2}\right)^{-4j}$.

To apply Theorem 3 first we must find values of the parameters, and we can easily see that

$$n = 3, \quad d = 4, \quad b_1 = 2h, \quad b_2 = -2j, \quad b_3 = 1;$$
$$\alpha_1 = \frac{r+\sqrt{ab}}{2}, \quad \alpha_2 = \frac{s+\sqrt{ac}}{2}, \quad \alpha_3 = \frac{\sqrt{c}(\sqrt{a}+\sqrt{b})}{\sqrt{b}(\sqrt{a}+\sqrt{c})}.$$

As in [16, Lemma 19], it can be easily shown that α_1, α_2 and α_3 are multiplicatively independent. Also, it is not difficult to see that $h(\alpha_1) = \frac{1}{2}\log\alpha_1$ and $h(\alpha_2) = \frac{1}{2}\log\alpha_2$. The minimal polynomial of α_3 is

$$\begin{aligned} p_3(X) = {} & b^2(c-a)^2X^4 - 4b^2c(c-a)X^3 + \\ & 2bc(3bc - a^2 - ac - ab)X^2 - 4bc^2(b-a)X + c^2(b-a)^2 \end{aligned}$$

divided by the greatest common divisor of its coefficients, which we will denote with g. The zeros of the polynomial $p_3(X)$ are $\beta_1 = \frac{\sqrt{c}(-\sqrt{a}+\sqrt{b})}{\sqrt{b}(\sqrt{a}+\sqrt{c})}$, $\beta_2 = \frac{\sqrt{c}(\sqrt{a}+\sqrt{b})}{\sqrt{b}(-\sqrt{a}+\sqrt{c})}$, $\beta_3 = \frac{\sqrt{c}(-\sqrt{a}+\sqrt{b})}{\sqrt{b}(-\sqrt{a}+\sqrt{c})}$ and α_3. It holds

$$\beta_1 < \beta_3 < 1$$

and

$$1 < \alpha_3 < \beta_2,$$

which implies

$$\begin{aligned} h(\alpha_3) &= \frac{1}{4}\left(\log\frac{b^2(c-a)^2}{g} + \log\alpha_3 + \log\beta_2\right) \\ &\leq \frac{1}{4}\left(\log(b^2(c-a)^2) + \log\alpha_3 + \log\beta_2\right). \end{aligned}$$

We can observe that

$$\begin{aligned} h(\alpha_3) &\leq \frac{1}{4}\left(\log(b^2(c-a)^2) + \log\frac{c(\sqrt{a}+\sqrt{b})^2}{b(c-a)}\right) \\ &= \frac{1}{4}\log(cb(c-a)(\sqrt{a}+\sqrt{b})^2) \\ &< \frac{1}{4}\log c^4 = \log c. \end{aligned}$$

Since the function on the right hand side of the inequality in Theorem 3 is decreasing in A_3 we can take

$$A_1 = 4\frac{1}{2}\log\alpha_1 = 2\log\alpha_1, \quad A_2 = 2\log\alpha_2, \quad A_3 = 4\log c = \log c^4.$$

Observe that $A_1 < A_2 < A_3$ and $j < h$, so we have $E = \max\left\{\frac{2h}{\log\alpha_1}, 3\right\}$. Since $0.66\sqrt{ac} > 0.66r > \log r^3$, which is true for every $r > 10$, we have $h > 0.66\sqrt{ac} > 3\log r > 3\log\alpha_1$ which implies $2h/\log\alpha_1 > 3$, i.e. we can take $E = 2h/\log\alpha_1$ and apply Theorem 3 to get,

$$\log|\Lambda_1| > -5.3n^{0.5-n}(n+1)^{n+1}(n+8)^2(n+5)31.44^n d^2 \log\frac{2h}{\log\alpha_1} \cdot 2\log\alpha_1 \cdot 2\log\alpha_2 \cdot 4\log c \cdot \log(3nd).$$

On the other hand, from Lemma 18 and the fact that $|b_1|A_1 < |b_2|A_2$ we have

$$\log|\Lambda_1| < -4j\log\alpha_2 < -4h\log\alpha_1,$$

which now implies

$$4h\log\alpha_1 < 5.3n^{0.5-n}(n+1)^{n+1}(n+8)^2(n+5)31.44^n d^2 \log\frac{2h}{\log\alpha_1} \cdot 2\log\alpha_1 \cdot 2\log\alpha_2 \cdot 4\log c \cdot \log(3nd).$$

We put $n = 3$, $d = 4$ and get

$$\frac{h}{\log 2h - \log\log\sqrt{10^5}} < 6.005175 \cdot 10^{11}\log\alpha_2\log c,$$

where we have used $\alpha_1 > \sqrt{ab} > \sqrt{10^5}$. Now we use that $\alpha_2 < \sqrt{ac+4}$, $c \le ac$ and since the left hand side of the inequality is increasing in h we can use $h > 0.666662\sqrt{ac}$ to get

$$ac < 1.08915 \cdot 10^{34} \tag{22}$$

and

$$h < 6.95745 \cdot 10^{16}. \tag{23}$$

We collect these observations in the next Proposition.

Proposition 5. *Let $\{a,b,c,d,e\}$ be a $D(4)$-quintuple such that $a<b<c<d<e$, then $ac<1.08915\cdot 10^{34}$ and $h<6.95745\cdot 10^{16}$. Moreover,*

$$\frac{h}{\log 2h-\log\log\sqrt{10^5}}<6.005175\cdot 10^{11}\log\alpha_2\log c.$$

To get a sharper bound on ac and h, which we need later, we will use the Proposition 5 together with a tool due to Mignotte [19] in combination with Laurent's theorem. Laurent's theorem is needed to resolve some cases in Mignotte's theorem. First, we will state Mignotte's theorem and show how can it be applied to $D(4)$-quintuples. We aim to give the most general algorithm to find appropriate parameters, so it can be clear how we can easily repeat the procedure multiple times to get better results.

Theorem 4 *(Mignotte). We observe three non-zero algebraic numbers α_1, α_2 and α_3, which are either all real and greater than 1 or all complex of modulus one and all different from 1. Moreover, we assume that either the three numbers α_1, α_2 and α_3 are multiplicatively independent, or two of these numbers are multiplicatively independent and the third one is a root of unity. Put*

$$\mathcal{D}=[\mathbb{Q}(\alpha_1,\alpha_2,\alpha_3):\mathbb{Q}]/[\mathbb{R}(\alpha_1,\alpha_2,\alpha_3):\mathbb{R}].$$

We also consider three positive coprime rational integers b_1,b_2,b_3, and the linear form

$$\Lambda=b_2\log\alpha_2-b_1\log\alpha_1-b_3\log\alpha_3,$$

where the logarithms of α_i are arbitrary determinations of the logarithm, but which are all real or all purely imaginary. And we assume also that

$$b_2|\log\alpha_2|=b_1|\log\alpha_1|+b_3|\log\alpha_3|\pm|\Lambda|.$$

We put

$$d_1=\gcd(b_1,b_2),\ d_3=\gcd(b_3,b_2),\ b_1=d_1b_1',\ b_2=d_1b_2'=d_3b_2'',\ b_3=d_3b_3''.$$

Let $\rho>e$ be a real number and put $\lambda=\log\rho$. Let a_1, a_2 and a_3 be real numbers such that

$$a_i\geq\rho|\log\alpha_i|-\log|\alpha_i|+2\mathcal{D}h(\alpha_i),\quad i=1,2,3,$$

and assume further that

$$\Omega:=a_1a_2a_3\geq 2.5,\quad \text{and}\quad A:=\min\{a_1,a_2,a_3\}\geq 0.62.$$

Let K, L and M be positive integers with

$$L \geq 4 + \mathcal{D}, \quad K = \lfloor M\Omega L \rfloor, \quad \textit{where } M \geq 3.$$

Let $0 < \chi \leq 2$ *be fixed. Define*

$$\begin{aligned} c_1 &= \max\left\{(\chi ML)^{2/3}, \sqrt{2ML/A}\right\}, \\ c_2 &= \max\left\{2^{1/3}(ML)^{2/3}, \sqrt{M/A}L\right\}, \\ c_3 &= (6M^2)^{1/3}L, \end{aligned}$$

and then put

$$\begin{aligned} R_1 &= \lfloor c_1 a_2 a_3 \rfloor, \quad S_1 = \lfloor c_1 a_1 a_3 \rfloor, \quad T_1 = \lfloor c_1 a_1 a_2 \rfloor, \\ R_2 &= \lfloor c_2 a_2 a_3 \rfloor, \quad S_2 = \lfloor c_2 a_1 a_3 \rfloor, \quad T_2 = \lfloor c_2 a_1 a_2 \rfloor, \\ R_3 &= \lfloor c_3 a_2 a_3 \rfloor, \quad S_3 = \lfloor c_3 a_1 a_3 \rfloor, \quad T_3 = \lfloor c_3 a_1 a_2 \rfloor. \end{aligned}$$

Let also

$$R = R_1 + R_2 + R_3 + 1, \quad S = S_1 + S_2 + S_3 + 1, \quad T = T_1 + T_2 + T_3 + 1.$$

Define

$$c_0 = \max\left\{\frac{R}{La_2a_3}, \frac{S}{La_1a_3}, \frac{T}{La_1a_2}\right\}.$$

Finally, assume that

$$\begin{aligned} &\left(\frac{KL}{2} + \frac{L}{4} - 1 - \frac{2K}{3L}\right)\lambda + 2\mathcal{D}\log 1.36 \\ &\qquad \geq (\mathcal{D}+1)\log L + 3gL^2c_0\Omega + \mathcal{D}(K-1)\log \widetilde{b} + 2\log K, \end{aligned} \tag{24}$$

where

$$g = \frac{1}{4} - \frac{K^2L}{12RST}, \quad b' = \left(\frac{b_1'}{a_2} + \frac{b_2'}{a_1}\right)\left(\frac{b_3''}{a_2} + \frac{b_2''}{a_3}\right), \quad \widetilde{b} = \frac{e^3c_0^2\Omega^2L^2}{4K^2} \times b'.$$

Then either

$$\log|\Lambda| > -(KL + \log(3KL))\lambda, \tag{25}$$

or (A1) *there exist two non-zero rational integers* r_0 *and* s_0 *such that*

$$r_0 b_2 = s_0 b_1$$

with

$$|r_0| \leq \frac{(R_1+1)(T_1+1)}{\mathcal{M}-T_1} \quad \textit{and} \quad |s_0| \leq \frac{(S_1+1)(T_1+1)}{\mathcal{M}-T_1}$$

where

$$\mathcal{M} = \max\{R_1+S_1+1, S_1+T_1+1, R_1+T_1+1, \chi\mathcal{V}\},$$
$$\mathcal{V} = \sqrt{(R_1+1)(S_1+1)(T_1+1)},$$

or (A2) *there exist rational integers r_1, s_1, t_1 and t_2, with $r_1 s_1 \neq 0$ such that*

$$(t_1 b_1 + r_1 b_3) s_1 = r_1 b_2 t_2, \quad \gcd(r_1, t_1) = \gcd(s_1, t_2) = 1,$$

which also satisfy

$$|r_1 s_1| \leq \delta \cdot \frac{(R_1+1)(S_1+1)}{\mathcal{M}-\max\{R_1, S_1\}},$$
$$|s_1 t_1| \leq \delta \cdot \frac{(S_1+1)(T_1+1)}{\mathcal{M}-\max\{S_1, T_1\}},$$
$$|r_1 t_2| \leq \delta \cdot \frac{(R_1+1)(T_1+1)}{\mathcal{M}-\max\{R_1, T_1\}},$$

where $\delta = \gcd(r_1, s_1)$. Moreover, when $t_1 = 0$ we can take $r_1 = 1$, and when $t_2 = 0$ we can take $s_1 = 1$.

We consider the linear form

$$\Lambda = -\Lambda_1 = 2j \log\alpha_2 - 2h \log\alpha_1 - \log\alpha_3.$$

It is important to notice that we have $c > b > 10^5$.

As before we have

$$\mathcal{D} = 4, \quad b_1 = 2h, \quad b_2 = 2j, \quad b_3 = 1,$$

and we can again take

$$h(\alpha_1) = \frac{1}{2}\log\alpha_1, \quad h(\alpha_2) = \frac{1}{2}\log\alpha_2, \quad h(\alpha_3) < \log c.$$

Observe that

$$\log\alpha_3 < \log\left(1+\sqrt{\frac{a}{b}}\right) < \log 2 < 0.694.$$

Now we have to choose $a_i \geq \rho|\log\alpha_i| - \log|\alpha_i| + 2\mathcal{D}h(\alpha_i)$ for $i \in \{1,2,3\}$. In each case we have $|\log\alpha_i| = \log|\alpha_i| = \log\alpha_i$. Let $i = 1$, then

$$a_1 \geq \rho \log \alpha_1 - \log \alpha_1 + 4 \cdot \log \alpha_1 = (\rho + 3) \log \alpha_1$$

and a similar observation is true for $i = 2$. For $i = 3$ we have

$$a_3 \geq \rho \log \alpha_3 - \log \alpha_3 + 2 \cdot 4 \cdot \log c,$$

so we see that we can take

$$\begin{aligned} a_1 &= (\rho + 3) \log \alpha_1 \\ a_2 &= (\rho + 3) \log \alpha_2 \\ a_3 &= 8(\log c + 0.08675(\rho - 1)). \end{aligned}$$

For the simplicity of the proof we will give intervals for parameters M, L and ρ, but we will not give their explicit values, because we will search within these intervals to find the values which give us the best possible bound on index h. From now on, when ever is needed, we assume that $\chi = 2$, $\rho \in [5.5, 14]$, $L \in [700, 1500]$, and $M \in [3, 10]$. These intervals were chosen since they seemed sufficient, after observing some random values, for finding an optimal value for upper bound on h and also because they satisfy all conditions needed, as we will prove.

Now, let us observe which conditions these parameters must satisfy so we can use Theorem 4.

It is easy to see that we always have $a_1 < a_2$, so $A = \min\{a_1, a_2, a_3\} = \min\{a_1, a_3\}$. If $A = a_1$ we have $A = (\rho + 3) \log \alpha_1 > 5 \log \sqrt{ab}$, and if $A = a_3$ then $A > 8 \log c$, so in either case it is $A \geq 0.62$. Moreover, it is also easy to see that we always have $\Omega = a_1 a_2 a_3 > 2.5$.

The values c_1, c_2 and c_3 can easily be calculated for specific values of the parameters.

We get an upper bound for c_0 after observing that

$$\frac{R}{L a_2 a_3} = \frac{R_1 + R_2 + R_3 + 1}{L a_2 a_3} < \frac{c_1 + c_2 + c_3 + 1}{L}$$

and since the same is true for S and T, we have $c_0 < (c_1 + c_2 + c_3 + 1)/L$.

Also

$$\Omega = a_1 a_2 a_3 = 8(\rho + 3)^2 \log \alpha_1 \log \alpha_2 (\log c + 0.08675(\rho - 1))$$

and

$$K = \lfloor M \Omega L \rfloor = \lfloor 8ML(\rho + 3)^2 \log \alpha_1 \log \alpha_2 (\log c + 0.08675(\rho - 1)) \rfloor.$$

To see when inequality (24) holds, let us observe it part by part:

We have $M\Omega L - 1 < K \le M\Omega L$ so

$$\begin{aligned}
&\left(\frac{KL}{2} + \frac{L}{4} - 1 - \frac{2K}{3L}\right)\lambda + 2\mathcal{D}\log 1.36 > \\
&\quad > M\Omega L\left(\frac{L}{2} - \frac{2}{3L}\right)\lambda - \left(\frac{L}{2} - \frac{2}{3L}\right)\lambda + \left(\frac{L}{4} - 1\right)\lambda + 2\mathcal{D}\log 1.36 \\
&\quad = 8ML(\rho+3)^2\left(\frac{L}{2} - \frac{2}{3L}\right)\lambda \log\alpha_1 \log\alpha_2 \log c \\
&\qquad + 8ML(\rho+3)^2\left(\frac{L}{2} - \frac{2}{3L}\right)\lambda \cdot 0.08675(\rho-1)\log\alpha_1\log\alpha_2 \\
&\qquad + \left(\frac{L}{4} - 1\right)\lambda + 2\mathcal{D}\log 1.36 - \left(\frac{L}{2} - \frac{2}{3L}\right)\lambda.
\end{aligned}$$

On the other hand, for the expressions on the right hand side of the inequality (24) it holds:

1. Since we can use $ac < 1.08915 \cdot 10^{34}$ we get a numerical value

$$(\mathcal{D}+1)\log L + 2\log K \le 5\log L + 2\log(8ML(\rho+3)^2\log^2\sqrt{ac+4}\log ac).$$

2. Also, from $g = 1/4 - (K^2L)/(12RST) < 1/4$ we get

$$\begin{aligned}
3gL^2c_0\Omega < \frac{3}{4}L^2c_0\Omega &\le \frac{3}{4}L^2c_0 \cdot 8(\rho+3)^2\log\alpha_1\log\alpha_2\log c \\
&\quad + \frac{3}{4}L^2c_0 \cdot 8 \cdot 0.08675(\rho-1)(\rho+3)^2\log\alpha_1\log\alpha_2.
\end{aligned}$$

3. To approximate the last part of the right hand side of the inequality, observe that from inequalities $\log\alpha_3 < 2\log\alpha_1$ and $\Lambda_1 > 0$ we have $2(h+1)\log\alpha_1 - 2j\log\alpha_2 > 0$, which gives us

$$\frac{b_2}{a_1} < \frac{b_1+2}{a_2}.$$

Also, since $2\log\alpha_2 > \log c$ and $\rho \ge 5.5$, we have $b_3/a_2 < 2/a_3$ and since $j < h$ we get

$$b' < \frac{(4h+2)(2h+2)}{8(\rho+3)\log\alpha_2\log c}.$$

Using $c > 10^5$, $h < 6.95745 \cdot 10^{16}$ and values of the parameters, we can calculate an upper bound for b'.
Then we have

$$\frac{K}{\Omega} > \frac{M\Omega L - 1}{\Omega} > ML - 1$$

and

$$\log \widetilde{b} < \log\left(\frac{c_0^2}{4}e^3\frac{1}{(ML-1)^2}L^2b'\right).$$

Finally,

$$\begin{aligned}\mathcal{D}(K-1)\log\widetilde{b} &< 4M\Omega L\log\widetilde{b}\\ &= 32ML(\rho+3)^2\log\widetilde{b}\log\alpha_1\log\alpha_2\log c\\ &+ 32ML(\rho+3)^2\log\widetilde{b}\cdot 0.08675(\rho-1)\log\alpha_1\log\alpha_2.\end{aligned}$$

As we can see from above, we have expressions of the form $\log\alpha_1\log\alpha_2$, $\log\alpha_1\log\alpha_2\times\log c$ and numerical values, and to see if some selected values of the parameters M, L and ρ satisfy inequality (24) it is enough to compare coefficients of these expressions. For each selection of values for the parameters M, L and ρ which satisfy these conditions, we can apply Theorem 4 and have that either cases *(A1)* or *(A2)* hold or inequality (25) holds. Let us first consider this inequality. We then have

$$\begin{aligned}\log|-\Lambda_1| &> -(KL+\log(3KL))\lambda\\ &\geq -(ML^2\Omega+\log(3ML^2\Omega))\log\rho,\end{aligned}$$

and on the other hand,

$$\log|-\Lambda_1| < -4j\log\alpha_2 < -4h\log\alpha_1$$

which holds since Lemma 18, so

$$4h\log\alpha_1 < (ML^2\Omega+\log(3ML^2\Omega))\log\rho.$$

Notice that $ML^2\Omega > 8ML^2(\rho+3)^2\log\sqrt{ab}\log\sqrt{ac}\log c > 3.81\cdot 10^{10}$, and for $x > 3.81\cdot 10^{10}$ we have $\log 3x < 6.7\cdot 10^{-10}x$, so we observe

$$4h\log\alpha_1 < ML^2\Omega(1+6.7\cdot 10^{-10})\log\rho,$$

i.e.

$$h < 2ML^2(\rho+3)^2\log\rho(1+6.7\cdot 10^{-10})\left(1+\frac{0.08675}{\log 10^5}(\rho-1)\right)\log\alpha_2\log c.$$

From now on, to shorten an expression x, with $G(x)$ we will denote the numerical value we get by inserting the values of the parameters χ, M, L, ρ, and later also the values of the parameters ϱ and μ, in the part of the expression for the upper bound of x which doesn't contain elements of a triple $\{a,b,c\}$. In this expression with $G(h^{(1)})$ we denote

$$G(h^{(1)}) := 2ML^2(\rho+3)^2\log\rho(1+6.7\cdot 10^{-10})\left(1+\frac{0.08675}{\log 10^5}(\rho-1)\right),$$

so we have $h < G(h^{(1)})\cdot\log\alpha_2\log c$.

If the inequality (25) does not hold, then one of the cases *(A1)* or *(A2)* holds.

Notice that $\mathcal{M} > \chi\mathcal{V} > \chi c_1^{3/2}a_1a_2a_3$. For each a_i we calculate the lower bounds

$$a_2 > a_1 > (\rho+3)\log 10^{5/2} := A_{1,2}, \qquad a_3 > 8(\log 10^5 + 0.08675(\rho-1)) := A_3.$$

Observe that since $a_2 > a_1$, it always holds $\max\{R_1, S_1\} = R_1$. On the other hand, the values of $\max\{S_1, T_1\}$ and $\max\{R_1, T_1\}$ depend on the values of a triple $\{a, b, c\}$, so we must address these cases separately.

Let us denote and observe

$$\begin{aligned} B_1 &:= \frac{(R_1+1)(S_1+1)}{\mathcal{M}-\max\{R_1,S_1\}} < \frac{(c_1a_2a_3+1)(c_1a_1a_3+1)}{\chi c_1^{3/2}a_1a_2a_3 - c_1a_2a_3} \\ &= \frac{1+\frac{1}{c_1a_2a_3}}{\frac{\chi}{2}-\frac{1}{2c_1^{1/2}a_1}}\left(0.5c_1^{1/2}+\frac{1}{2c_1^{1/2}a_1a_3}\right)a_3 \\ &< \frac{1+\frac{1}{c_1A_{1,2}A_3}}{\frac{\chi}{2}-\frac{1}{2c_1^{1/2}A_{1,2}}}\left(0.5c_1^{1/2}+\frac{1}{2c_1^{1/2}A_{1,2}A_3}\right)8\left(1+\frac{0.08675}{\log 10^5}(\rho-1)\right)\log c \\ &=: G(B_1)\cdot\log c. \end{aligned}$$

Let us assume that $\max\{S_1, T_1\} = S_1$. Then

$$\begin{aligned} B_2 &:= \frac{(S_1+1)(T_1+1)}{\mathcal{M}-\max\{S_1,T_1\}} < \frac{(c_1a_1a_3+1)(c_1a_1a_2+1)}{\chi c_1^{3/2}a_1a_2a_3 - c_1a_1a_3} \\ &= \frac{1+\frac{1}{c_1a_1a_3}}{\frac{\chi}{2}-\frac{1}{2c_1^{1/2}a_2}}\left(0.5c_1^{1/2}+\frac{1}{2c_1^{1/2}a_2^2}\right)a_2 \\ &< \frac{1+\frac{1}{c_1A_{1,2}A_3}}{\frac{\chi}{2}-\frac{1}{2c_1^{1/2}A_{1,2}}}\left(0.5c_1^{1/2}+\frac{1}{2c_1^{1/2}A_{1,2}^2}\right)(\rho+3)\log\alpha_2 \\ &=: G(B_2^{(1)})\cdot\log\alpha_2. \end{aligned}$$

On the other hand, if $\max\{S_1, T_1\} = T_1$, then

$$\begin{aligned} B_2 &= \frac{(S_1+1)(T_1+1)}{\mathcal{M}-\max\{S_1,T_1\}} < \frac{(c_1a_1a_3+1)(c_1a_1a_2+1)}{\chi c_1^{3/2}a_1a_2a_3 - c_1a_1a_2} \\ &= \frac{1+\frac{1}{c_1a_1a_2}}{\frac{\chi}{2}-\frac{1}{2c_1^{1/2}a_3}}\left(0.5c_1^{1/2}+\frac{1}{2c_1^{1/2}a_2a_3}\right)a_2 \end{aligned}$$

$$< \frac{1+\frac{1}{c_1A_{1,2}^2}}{\frac{\chi}{2}-\frac{1}{2c_1^{1/2}A_3}}\left(0.5c_1^{1/2}+\frac{1}{2c_1^{1/2}A_{1,2}A_3}\right)(\rho+3)\log\alpha_2$$
$$=: G(B_2^{(2)})\cdot\log\alpha_2,$$

where we gave these expressions in the form where it is clear that they are decreasing in variables a_1, a_2 and a_3, so we can use lower bounds of these variables to get an upper bound on B_2. Observe that

$$G(B_2^{(1)}) = \frac{(c_1A_{1,2}A_3+1)(c_1A_{1,2}^2+1)}{\chi c_1^{3/2}A_{1,2}^2A_3-c_1A_{1,2}A_3},\ G(B_2^{(2)}) = \frac{(c_1A_{1,2}A_3+1)(c_1A_{1,2}^2+1)}{\chi c_1^{3/2}A_{1,2}^2A_3-c_1A_{1,2}^2},$$

and since these expressions only differ in their denominators, it is easy to see that if $A_3 > A_{1,2}$, then $G(B_2^{(1)}) > G(B_2^{(2)})$. Inequality $A_3 > A_{1,2}$ will hold for $\rho\in[5.5,14]$, which is a reason why we have chosen that interval for our observations.

Now we define $G(B_2) = \max\{G(B_2^{(1)}), G(B_2^{(2)})\}$, so

$$B_2 < G(B_2)\cdot\log\alpha_2.$$

Similarly, first we will assume that $\max\{R_1,T_1\} = R_1$, so

$$B_3 := \frac{(R_1+1)(T_1+1)}{\mathcal{M}-\max\{R_1,T_1\}} < \frac{(c_1a_2a_3+1)(c_1a_1a_2+1)}{\chi c_1^{3/2}a_1a_2a_3-c_1a_2a_3}$$
$$= \frac{1+\frac{1}{c_1a_2a_3}}{\frac{\chi}{2}-\frac{1}{2c_1^{1/2}a_1}}\left(0.5c_1^{1/2}+\frac{1}{2c_1^{1/2}a_1a_2}\right)a_2$$
$$< \frac{1+\frac{1}{c_1A_{1,2}A_3}}{\frac{\chi}{2}-\frac{1}{2c_1^{1/2}A_{1,2}}}\left(0.5c_1^{1/2}+\frac{1}{2c_1^{1/2}A_{1,2}^2}\right)(\rho+3)\log\alpha_2$$
$$=: G(B_3^{(1)})\cdot\log\alpha_2,$$

and if $\max\{R_1,T_1\} = T_1$, then

$$B_3 = \frac{(R_1+1)(T_1+1)}{\mathcal{M}-\max\{R_1,T_1\}} < \frac{(c_1a_2a_3+1)(c_1a_1a_2+1)}{\chi c_1^{3/2}a_1a_2a_3-c_1a_1a_2}$$
$$= \frac{1+\frac{1}{c_1a_1a_2}}{\frac{\chi}{2}-\frac{1}{2c_1^{1/2}a_3}}\left(0.5c_1^{1/2}+\frac{1}{2c_1^{1/2}a_2a_3}\right)a_2$$
$$< \frac{1+\frac{1}{c_1A_{1,2}^2}}{\frac{\chi}{2}-\frac{1}{2c_1^{1/2}A_3}}\left(0.5c_1^{1/2}+\frac{1}{2c_1^{1/2}A_{1,2}A_3}\right)(\rho+3)\log\alpha_2$$
$$=: G(B_3^{(2)})\cdot\log\alpha_2.$$

Analogously, $G(B_3) = \max\{G(B_3^{(1)}), G(B_3^{(2)})\}$ and

$$B_3 < G(B_3) \cdot \log \alpha_2.$$

Notice that since we have chosen the same lower bounds on a_1 and a_2, we have $G(B_2^{(1)}) = G(B_3^{(1)})$ and $G(B_2^{(2)}) = G(B_3^{(2)})$, and also $G(B_2) = G(B_3)$.

Now, let us consider the case *(A2)*. Here we have some integers r_1, s_1, t_1 and t_2, such that

$$(t_1 b_1 + r_1 b_3)s_1 = r_1 b_2 t_2, \quad \gcd(r_1, t_1) = \gcd(s_1, t_2) = 1,$$

and

$$|r_1 s_1| \le \delta B_1, \quad |s_1 t_1| \le \delta B_2, \quad |r_1 t_2| \le \delta B_3, \quad \delta = \gcd(r_1, s_1).$$

We have $r_1 = \delta r_1'$ and $s_1 = \delta s_1'$. Since $b_1 = 2h$, $b_2 = 2j$ and $b_3 = 1$ we also have

$$s_1' t_1 \cdot 2h + \delta r_1' s_1' = r_1' t_2 \cdot 2j,$$

and

$$|\delta r_1' s_1| \le B_1, \quad |s_1' t_1| \le B_2, \quad |r_1' t_2| \le B_3.$$

First, let us consider the case when $t_2 = 0$. Then $\gcd(s_1, t_2) = s_1 = 1$ and from $(t_1 b_1 + r_1 b_3)s_1 = 0$, since $s_1 \neq 0$, we get $t_1 b_1 = -r_1 b_3$, i.e. $2ht_1 = -r_1$. Since $\gcd(r_1, t_1) = 1$, we conclude that $t_1 = \mp 1$ and $r_1 = \pm 2h$. Also, we see from the observations stated before that

$$|r_1 s_1| = 2h \le B_1 < \frac{\left(c_1 A_{1,2} + \frac{1}{A_3}\right)(c_1 A_{1,2} A_3 + 1)}{\chi c_1^{3/2} A_{1,2}^2 A_3 - c_1 A_{1,2} A_3} a_3.$$

Since $\chi = 2$ and $A = \min\{a_1, a_3\} > 1$, we have that

$$c_1 = \max\left\{(\chi ML)^{2/3}, \sqrt{2ML/A}\right\} = (2ML)^{2/3}.$$

If we use minimal and maximal values of our parameters M i L, we get

$$260 < c_1 < 966.$$

Using these values and lower bounds $A_{1,2} > 48.9$, $A_3 > 8 \log 10^5 > 92.1$ and the fact that $a_3 < 8(1 + 13 \cdot 0.08675/\log 10^5) \log c$, we get the inequality

$$B_1 < 979.86 \log c.$$

So, we see that the inequality $2h < 979.86 \log c$ holds. From Proposition 17 we have that $h > 0.666662\sqrt{ac} \geq 0.666662\sqrt{c}$, which implies

$$\sqrt{c} < 734.91 \log c.$$

Solving this inequality, we get $c < 1.9701 \cdot 10^8$. We will see that this upper bound is much lower than the upper bound we will get in the case that $t_2 \neq 0$.

Now, let us assume that $t_2 \neq 0$. We can multiply the linear form Λ_1 with factor $r_1' t_2 \neq 0$, and after rearranging we get a linear form in two logarithms

$$r_1' t_2 \Lambda_1 = 2h \log\left(\alpha_1^{r_1' t_2} \cdot \alpha_2^{-s_1' t_1}\right) - \log\left(\alpha_2^{\delta r_1' s_1'} \cdot \alpha_3^{-r_1' t_2}\right), \tag{26}$$

where $\delta = \gcd(r_1, s_1)$, $r_1' = r_1/\delta$ and $s_1' = s_1/\delta$. We would like to apply the next result from [17] to this linear form in two logarithms.

Theorem 5 *(Laurent). Let a_1', a_2', h', ϱ and μ be real numbers with $\varrho > 1$ and $1/3 \leq \mu \leq 1$. Set*

$$\sigma = \frac{1 + 2\mu - \mu^2}{2}, \quad \lambda' = \sigma \log \varrho, \quad H = \frac{h'}{\lambda'} + \frac{1}{\sigma},$$
$$\omega = 2\left(1 + \sqrt{1 + \frac{1}{4H^2}}\right), \quad \theta = \sqrt{1 + \frac{1}{4H^2}} + \frac{1}{2H}.$$

Consider the linear form

$$\Lambda = b_2 \log \gamma_2 - b_1 \log \gamma_1,$$

where b_1 and b_2 are positive integers. Suppose that γ_1 are γ_2 multiplicatively independent. Put $D = [\mathbb{Q}(\gamma_1, \gamma_2) : \mathbb{Q}]/[\mathbb{R}(\gamma_1, \gamma_2) : \mathbb{R}]$, and assume that

$$h' \geq \max\left\{D\left(\log\left(\frac{b_1}{a_2'} + \frac{b_2}{a_1'}\right) + \log \lambda' + 1.75\right) + 0.06, \lambda', \frac{D \log 2}{2}\right\},$$
$$a_i' \geq \max\{1, \varrho|\log \gamma_i| - \log|\gamma_i| + 2Dh(\gamma_i)\}, \quad i = 1, 2,$$
$$a_1' a_2' \geq \lambda'^2.$$

Then

$$\log|\Lambda| \geq -C\left(h' + \frac{\lambda'}{\sigma}\right)^2 a_1' a_2' - \sqrt{\omega\theta}\left(h' + \frac{\lambda'}{\sigma}\right) - \log\left(C'\left(h' + \frac{\lambda'}{\sigma}\right)^2 a_1' a_2'\right)$$

with

$$C = \frac{\mu}{\lambda'^3\sigma}\left(\frac{\omega}{6} + \frac{1}{2}\sqrt{\frac{\omega^2}{9} + \frac{8\lambda'\omega^{5/4}\theta^{1/4}}{3\sqrt{a_1'a_2'}H^{1/2}} + \frac{4}{3}\left(\frac{1}{a_1'} + \frac{1}{a_2'}\right)\frac{\lambda'\omega}{H}}\right)^2,$$

$$C' = \sqrt{\frac{C\sigma\omega\theta}{\lambda'^3\mu}}.$$

To apply Theorem 5 on the linear form (26) first we must check that the conditions of the theorem are satisfied. Since α_1, α_2 and α_3 are multiplicatively independent, so are γ_1 and γ_2.

Since we have from the inequality (25) that $h < G(h^{(1)}) \cdot \log\alpha_2 \log c$, it is sufficient to assume that $h \geq G(h^{(1)}) \cdot \log\alpha_2 \log c$ and aim to find the best possible upper bound on h in this case.

Notice that,

$$\begin{aligned} h(\gamma_1) &\leq 0.5B_1 \log\alpha_2 + B_2 \log c \\ &< (0.5G(B_1) + G(B_2)) \log\alpha_2 \log c =: G(h(\gamma_1)) \cdot \log\alpha_2 \log c, \\ h(\gamma_2) &\leq 0.5B_2 \log\alpha_1 + 0.5B_3 \log\alpha_2 \\ &< B_3 \log\alpha_2 < G(B_3) \cdot \log^2\alpha_2 =: G(h(\gamma_2)) \cdot \log^2\alpha_2, \\ |\log\gamma_1| &\leq B_1 \log\alpha_2 + 0.694B_3 \\ &\leq \left(G(B_1) + 0.694\frac{G(B_3)}{\log 10^5}\right) \log\alpha_2 \log c =: G(|\log\gamma_1|) \cdot \log\alpha_2 \log c \end{aligned}$$

and

$$\begin{aligned} |\log\gamma_2| &< \frac{B_2 + |\log\gamma_1|}{2h} < \frac{G(B_2)\log\alpha_2 + G(|\log\gamma_1|) \cdot \log\alpha_2 \log c}{2h} \\ &< \frac{\left(\frac{G(B_2)}{\log 10^5} + G(|\log\gamma_1|)\right) \log\alpha_2 \log c}{2G(h^{(1)}) \cdot \log\alpha_2 \log c} \\ &= \frac{\frac{G(B_2)}{\log 10^5} + G(|\log\gamma_1|)}{2G(h^{(1)})} =: G(|\log\gamma_2|). \end{aligned}$$

Now we would like to find which condition must parameters ϱ and μ satisfy in order to apply Theorem 5 and to get the lowest possible upper bound on h. First we must choose a_i', $i = 1, 2$, such that

$$a_i' \geq |\log\gamma_i|(\varrho + 1) + 8h(\gamma_i), \quad i = 1, 2.$$

We see that we can set

$$a_1' = (G(|\log\gamma_1|)(\varrho + 1) + 8G(h(\gamma_1))) \log\alpha_2 \log c =: G(a_1') \log\alpha_2 \log c,$$

and

$$a_2' = \left(\frac{G(|\log\gamma_2|)}{\log^2 10^{5/2}}(\varrho+1) + 8G(h(\gamma_2))\right)\log^2\alpha_2 =: G(a_2')\log^2\alpha_2.$$

We have

$$\frac{b_1}{a_2'} + \frac{b_2}{a_1'} \leq \frac{\frac{2}{G(a_2')} + \frac{2h}{G(a_1')}}{\log\alpha_2\log c} \leq \frac{h\left(\frac{2}{210.81\cdot G(a_2')} + \frac{2}{G(a_1')}\right)}{\log\alpha_2\log c}$$

where we used that since $c > 10^5$ then $h > 0.666662\sqrt{ac} > 0.666662\sqrt{10^5} > 210.81$. Denote

$$G(F) = \frac{2}{210.81\cdot G(a_2')} + \frac{2}{G(a_1')}$$

and

$$F := \frac{G(F)\cdot h}{\log\alpha_2\log c}.$$

Since we will observe only values $\varrho \leq 100$, and since $(D\log 2)/2 = 2\log 2 < 1.4$ and $\lambda' < \frac{3}{2}\log\varrho < 7$ we can take

$$h' = 4(\log F + \log\lambda') + 7.06.$$

Since we assumed that $h \geq G(h^{(1)})\log\alpha_2\log c$, we now have $F > G(F)\cdot G(h^{(1)})$ which implies

$$H = \frac{h'}{\lambda'} + \frac{1}{\sigma} > \frac{4\log(G(F)\cdot G(h^{(1)}))}{\lambda'} + \frac{1}{\sigma}.$$

Using this, for specific values of the parameters ϱ and μ we can calculate ω, θ, C and C' and by Theorem 5 we have

$$\log|r_1't_2\Lambda_1| > -C\left(h' + \frac{\lambda'}{\sigma}\right)^2 a_1'a_2' - \sqrt{\omega\theta}\left(h' + \frac{\lambda'}{\sigma}\right) - \log\left(C'\left(h' + \frac{\lambda'}{\sigma}\right)^2 a_1'a_2'\right).$$

Assume that $C' \leq 3C$ (which will be true in all our cases), then $\log 3x < 10^{-3}x$ holds for $x \geq 10343$, and in all our cases we will have $a_1'a_2' > 10343$ and also $\sqrt{\omega\theta} < 3$. Since we also have $\left(h' + \frac{\lambda'}{\sigma}\right) > 1$, we can observe the inequality

$$\log|r_1't_2\Lambda_1| > -C\cdot G(a_1')\cdot G(a_2')(1.001 + 3\cdot 10^{-4}C^{-1})\left(h' + \frac{\lambda'}{\sigma}\right)^2(\log\alpha_2)^3\log c.$$

We wish to find a minimal positive real number k for which the inequality $\log\alpha_2 < k\cdot\log\alpha_1$ holds. If we use that $\sqrt{ac} < \alpha_2$ and $\alpha_1 < \sqrt{ab+4}$ we get $ac < (ab+4)^k$. From Proposition 1 we have $ac < 237.952b^3$, and since $b > 10^5$ we find that the inequality holds for $k = 3.4753$.

Now, we see that we also have

$$\log|r_1' t_2 \Lambda_1| < \log B_3 - 4j\log\alpha_2 < \log B_3 - 4h\log\alpha_1,$$

and since $\log\alpha_2 < 3.4753\log\alpha_1$, hence

$$\begin{aligned} h &< \frac{3.4753}{4}\left(C\cdot G(a_1')\cdot G(a_2')(1.001 + 3\cdot 10^{-4}C^{-1}) + \frac{\log B_3}{\log 10^5(\log 10^{5/2})^3}\right)\cdot \\ &\quad \cdot\left(h' + \frac{\lambda'}{\sigma}\right)^2 \log^2\alpha_2\log c \\ &=: G(h^{(2)})\left(h' + \frac{\lambda'}{\sigma}\right)^2 \log^2\alpha_2\log c. \end{aligned}$$

Multiplying this expression with $\frac{G(F)}{\log\alpha_2\log c}$ yields

$$F < G(h^{(2)})\cdot G(F)\left(4\log F + 4\log\lambda' + 7.06 + \frac{\lambda'}{\sigma}\right)^2\log\alpha_2$$

and if we insert $\log\alpha_2 < \log\sqrt{ac+4}$ and an upper bound for ac we will get an upper bound for F, denote it with F_1, i.e. $F < F_1$. Now from the definition of F we have

$$h < \frac{F_1}{G(F)}\log\alpha_2\log c,$$

which gives us an upper bound on h and our goal is to minimize the numerical value $F_1/G(F)$.

As in [16], it is not difficult to see that in the case *(A1)* one obtains smaller values than in the case *(A2)* and therefore smaller upper bounds, so we see it is not necessary to calculate it.

Now, it remained to implement the described algorithm for the inequality (25) and the case *(A2)*. We observed these values of the parameters, $\chi = 2$ fixed, $\rho\in[5.5, 14]$ with step 0.5, $L\in[700, 1500]$ with step 1, $M\in[3, 10]$ with step 0.1 and after calculating the upper bound on h by Theorem 4, we also consider all values $\varrho\in[40, 85]$ with step 1 and $\mu\in[0.44, 0.76]$ with step 0.01 such that the coefficient $G(h^{(2)})$ is the least possible one.

In the first turn we used $ac < 1.08915\cdot 10^{34}$ and $h < 6.95745\cdot 10^{16}$, and the best value was obtained for the parameters $\rho = 11.5$, $M = 4.7$ and $L = 1043$ where we got $h < 5.66642\cdot 10^9\log\alpha_2\log c$, and for $\varrho = 59$ and $\mu = 0.63$ we got $h < 4.85941\cdot 10^{10}\log\alpha_2\log c$ in the case *(A2)*. From this we have $ac < 2.42372\cdot 10^{28}$ and $h < 1.03788\cdot 10^{14}$.

Now these new upper bounds can be used for the second turn and the best value is obtained for the parameters $\rho = 11$, $M = 4.6$, $L = 901$ where we got $h < 4.13857 \cdot 10^9 \log \alpha_2 \log c$, and for $\varrho = 59$, $\mu = 0.63$ we got $h < 3.53075 \cdot 10^{10} \log \alpha_2 \log c$. From this we obtain $ac < 1.22705 \cdot 10^{28}$ and $h < 7.38475 \cdot 10^{13}$.

We repeat the process three more times, and finally get that $ac < 1.17732 \cdot 10^{28}$, $h < 3.46289 \cdot 10^{10} \log \alpha_2 \log c$ and $h < 7.23357 \cdot 10^{13}$. This upper bound will be good enough for final steps of the proof so we state the next proposition.

Proposition 6. *Let $\{a, b, c, d, e\}$ be a $D(4)$-quintuple, such that $a < b < c < d < e$. Then $ac < 1.17732 \cdot 10^{28}$. Also, $h < 3.46289 \cdot 10^{10} \log \alpha_2 \log c$ and $h < 7.23357 \cdot 10^{13}$.*

8. $D(4)$-quintuples with regular triples

Let $\{a, b, c, d, e\}$ be a $D(4)$-quintuple with $a < b < c < d < e$. We have seen that $d = a + b + c + \frac{1}{2}(abc + rst)$ and

$$ad + 4 = x^2, \quad bd + 4 = y^2, \quad cd + 4 = z^2,$$

$$x = \frac{at + rs}{2}, \quad y = \frac{rt + bs}{2}, \quad z = \frac{cr + st}{2}.$$

If $\{a, b, c\}$ is a regular triple, i.e. $c = a + b + 2r$, then we also have $s = a + r$, $t = b + r$ and $d = rst$ and by a simple calculation we can see that

$$x = rs - 2, \quad y = rt - 2, \quad z = st + 2.$$

These relations will be helpful in proving some special claims about $D(4)$-quintuples with $c = a + b + 2r$.

Lemma 19. *If $\{a, b, c, d, e\}$ is a $D(4)$-quintuple, $a < b < c < d < e$, such that $c = a+b+2r$, then $2n > r$.*

Proof. From Lemma 15 we have

$$a\varepsilon l^2 + xl \equiv c\varepsilon n^2 + zn \pmod{d}.$$

Assume that equality holds, i.e. $a\varepsilon l^2 + xl = c\varepsilon n^2 + zn$. Since $x^2 = ad + 4$ and $z^2 = cd + 4$, if we multiply the equality by $zn + xl$ we get

$$(al^2 - cn^2)(\varepsilon(zn + xl) + d) = 4(n^2 - l^2),$$

which implies $al^2 - cn^2 | 4(n^2 - l^2)$. By Lemma 12 we have $n < l \leq 2n$ (it is easy to check that $n = l$ cannot hold), so we have $n \neq l$ and $1/2 \leq n/l < 1$, which gives us $al^2 - cn^2 \leq 4(l^2 - n^2)$ and

$$\left|\frac{a}{c}-\left(\frac{n}{l}\right)^2\right|\le\frac{4}{c}\left(1-\left(\frac{n}{l}\right)^2\right).$$

Since $c=a+b+2r>a+a+2a=4a$, we also have $a/c<1/4\le(n/l)^2$, so

$$\frac{1}{4}-\frac{a}{c}<\left(\frac{n}{l}\right)^2-\frac{a}{c}\le\frac{4}{c}\left(1-\left(\frac{n}{l}\right)^2\right)\le\frac{3}{c},$$

i.e. it must be $c<4a+12$. But, by Lemma 4 we have $b\ge a+57\sqrt{a}$ which would then imply

$$4a+57\sqrt{a}<a+b+2r<4a+12,$$

and this leads to a contradiction since $a\ge 1$. We can now conclude that our assumption was wrong, equality does not hold, so we have

$$d<|al^2-xl-cn^2+zn|\le|al^2-cn^2|+|xl-zn|.$$

It can be easily seen that $|al^2-cn^2|<cn^2$ and $|xl-zn|<zn$, so we have that $d<cn^2+zn$. Assume that $n\le r/2$. Since $d=rst=r(a+r)(b+r)$ and $z=st+2=(a+r)(b+r)+2$, we have

$$r(a+r)(b+r)<(a+b+2r)\frac{r^2}{4}+((a+r)(b+r)+2)\frac{r}{2}$$

and after canceling and rearranging we see that this cannot be true. We can now conclude $n>r/2$. □

The next Lemma can be proved similarly as [16, Lemma 19] so we omit a proof.

Lemma 20. *Let $\{a,b,c,d,e\}$ be a $D(4)$-quintuple such that $a<b<c<d<e$ and $c=a+b+2r$. Then*

$$8l\equiv 2(1-(-1)^j)(-\varepsilon c)\pmod{s},\quad 8n\equiv 2(1-(-1)^j)\varepsilon a\pmod{s}$$
$$8m\equiv 2(1-(-1)^k)(-\varepsilon c)\pmod{t},\quad 8n\equiv 2(1-(-1)^k)\varepsilon b\pmod{t},$$

where $\varepsilon=\pm1$.

Lemma 21. *Let $\{a,b,c,d,e\}$ be a $D(4)$-quintuple such that $a<b<c<d<e$ and $c=a+b+2r$. Then, at least one of the following congruences holds*

i) $8l\equiv 8n\equiv 0\pmod{s}$,
ii) $8m\equiv 8n\equiv 0\pmod{t}$,
iii) $8n\equiv -4\varepsilon r\left(\bmod \frac{st}{\gcd(s,t)}\right)$, *and* $\gcd(s,t)\in\{1,2,4\}$.

Proof. If j is even, then $1-(-1)^j=0$ implies $8l\equiv 8n\equiv 0 \pmod{s}$ and $i)$ holds.

If k is even, then $1-(-1)^k=0$ implies $8m\equiv 8n\equiv 0 \pmod{t}$ and $ii)$ holds.

If both j and k are odd, then

$$8l\equiv 4(-\varepsilon c) \pmod{s},\quad 8n\equiv 4\varepsilon a \pmod{s},$$
$$8m\equiv 4(-\varepsilon c) \pmod{t},\quad 8n\equiv 4\varepsilon b \pmod{t}.$$

From $s=a+r$ and $t=b+r$ we have $a\equiv -r \pmod{s}$ and $b\equiv -r \pmod{t}$, so

$$8n\equiv -4\varepsilon r \pmod{s},\quad 8n\equiv -4\varepsilon r \pmod{t},$$

i.e.

$$8n\equiv -4\varepsilon r \left(\operatorname{mod} \frac{st}{\gcd(s,t)}\right).$$

Since $c=s+t$ we can see that $\gcd(s,t)=\gcd(s,s+t)=\gcd(s,c)$, and from $ac+4=s^2$ we conclude $\gcd(s,c)|4$, which proves the statement of the lemma. □

We would like to use these results to obtain some effective bounds on elements $\{a,b,c\}$ in order to use Baker–Davenport reduction.

Set

$$\beta_1=\frac{x+\sqrt{ad}}{2},\quad \beta_2=\frac{y+\sqrt{bd}}{2},\quad \beta_3=\frac{z+\sqrt{cd}}{2}$$
$$\beta_4=\frac{\sqrt{c}(\varepsilon\sqrt{a}+\sqrt{d})}{\sqrt{a}(\varepsilon\sqrt{c}+\sqrt{d})},\quad \beta_5=\frac{\sqrt{c}(\varepsilon\sqrt{b}+\sqrt{d})}{\sqrt{b}(\varepsilon\sqrt{c}+\sqrt{d})},$$

and consider the following linear forms in logarithms

$$\Lambda_2=2l\log\beta_1-2n\log\beta_3+\log\beta_4,$$
$$\Lambda_3=2m\log\beta_2-2n\log\beta_3+\log\beta_5.$$

From [12] we have the next lemma, and to avoid confusion, we would like to emphasize that v_m and w_n here denote sequences connected to the extension of a triple to a quadruple, as in Section 3.

Lemma 22 *(Lemma 10 in [12]). Let $\{a,b,c,d\}$ be a $D(4)$-quadruple. If $v_m=w_n$, $m,n\neq 0$, then*

$$0<m\left(\frac{s+\sqrt{ac}}{2}\right)-n\log\left(\frac{t+\sqrt{bc}}{2}\right)+\log\frac{\sqrt{b}(x_0\sqrt{c}+z_0\sqrt{a})}{\sqrt{a}(y_1\sqrt{c}+z_1\sqrt{b})}$$
$$<2ac\left(\frac{s+\sqrt{ac}}{2}\right)^{-2m}.$$

We apply this lemma to $D(4)$-quadruples $\{a, b, d, e\}$ and $\{b, c, d, e\}$ to get upper bounds on Λ_2 and Λ_3.

Lemma 23. $0 < \Lambda_2 < 2ad\beta_1^{-4l}$ *and* $0 < \Lambda_3 < 2bd\beta_2^{-4m}$.

Now we will consider each case of Lemma 21 to get upper bounds on some elements of a $D(4)$-quintuple.

Lemma 24. *If* $8l \equiv 8n \equiv 0 \pmod{s}$, *then* $s \leq 201884$.

Proof. It is easy to see that $l \equiv n \equiv 0 \left(\bmod \frac{s}{\gcd(s,8)}\right)$, so $l = \frac{s}{\gcd(s,8)} l_1$ and $n = \frac{s}{\gcd(s,8)} n_1$ for some $l_1, n_1 \in \mathbb{N}$. Denote $s' = \frac{s}{\gcd(s,8)}$. We have

$$\Lambda_2 = 2s'l_1 \log \beta_1 - 2s'n_1 \log \beta_3 + \log \beta_4 = \log \beta_4 - 2s' \log \frac{\beta_3^{n_1}}{\beta_1^{l_1}}.$$

We can take

$$D = 4, \quad b_1 = 2s', \quad b_2 = 1, \quad \gamma_1 = \frac{\beta_3^{n_1}}{\beta_1^{l_1}}, \quad \gamma_2 = \beta_4.$$

As before, it is not hard to check that γ_1 and γ_2 are multiplicatively independent. The conjugates of γ_1 are

$$\frac{\beta_3^{n_1}}{\beta_1^{l_1}}, \quad \frac{\beta_3^{-n_1}}{\beta_1^{l_1}}, \quad \frac{\beta_3^{n_1}}{\beta_1^{-l_1}}, \quad \frac{\beta_3^{-n_1}}{\beta_1^{-l_1}}$$

and depending on whether $\beta_3^{n_1} > \beta_1^{l_1}$ or $\beta_3^{n_1} < \beta_1^{l_1}$ we have

$$h(\gamma_1) = \frac{1}{4}\left(\left|\log \frac{\beta_3^{n_1}}{\beta_1^{l_1}}\right| + \left|\log \frac{\beta_3^{n_1}}{\beta_1^{-l_1}}\right|\right) = \frac{n_1}{2} \log \beta_3$$

or

$$h(\gamma_1) = \frac{1}{4}\left(\left|\log \frac{\beta_3^{-n_1}}{\beta_1^{-l_1}}\right| + \left|\log \frac{\beta_3^{n_1}}{\beta_1^{-l_1}}\right|\right) = \frac{l_1}{2} \log \beta_1.$$

By Lemma 23

$$0 < \log \beta_4 - 2s' \log \frac{\beta_3^{n_1}}{\beta_1^{l_1}} < 2ad\beta_1^{-4l},$$

so we have

$$\left|\log \frac{\beta_3^{n_1}}{\beta_1^{l_1}}\right| < \frac{1}{2s'}(\log \beta_4 + 2ad\beta_1^{-4l}) < \frac{1}{2s'}\left(\log \beta_4 + \frac{2}{ad}\right).$$

It also holds

$$\beta_4 = \sqrt{\frac{c}{a}}\left(1 - \varepsilon\frac{\sqrt{c}-\sqrt{a}}{\sqrt{d}+\varepsilon\sqrt{c}}\right) \le \sqrt{\frac{c}{a}}\left(1 + \frac{\sqrt{c}}{\sqrt{d}-\sqrt{c}}\right) < 2\sqrt{\frac{c}{a}},$$

which implies

$$\left|\log\frac{\beta_3^{n_1}}{\beta_1^{l_1}}\right| < \frac{\log 2\sqrt{\frac{c}{a}}}{2s'} + \frac{2}{2s'ad} < \frac{\log 2s}{2s'} + \frac{1}{s'ad} = \gcd(s,8)\left(\frac{\log 2s}{2s} + \frac{1}{sad}\right).$$

We may assume $r > 10^4$, otherwise $s = a + r < 2r < 20000$, so we also have $s > 10^4$ and $d = rst > r^3 > 10^{12}$. Now we see

$$\left|\log\frac{\beta_3^{n_1}}{\beta_1^{l_1}}\right| < \gcd(s,8)\left(\frac{\log(2\cdot 10^4)}{2\cdot 10^4} + \frac{1}{10^4\cdot 10^{12}}\right) < 5\cdot 10^{-4}\gcd(s,8) < 0.004.$$

Also

$$\left|\frac{n_1}{2}\log\beta_3 - \frac{l_1}{2}\log\beta_1\right| < 0.002$$

and

$$h(\gamma_1) < \frac{l_1}{2}\log\beta_1 + 0.002.$$

The absolute values of the conjugates of $\gamma_2 = \beta_4$ are all greater than 1 and a minimal polynomial can be calculated analogously as for α_3 from the previous section so we have

$$h(\gamma_2) \le \frac{1}{4}\log\left(a^2(d-c)^2\cdot\frac{c^2}{a^2}\cdot\frac{(d-a)^2}{(d-c)^2}\right) < \frac{1}{2}\log(cd) < \log\beta_3.$$

Now, we can apply Theorem 5 and choose for the parameters $\varrho = 61$ and $\mu = 0.7$. We have $\sigma = 0.955$ and $3.92 < \lambda' < 3.93$ and take

$$a_1' := 4l_1\log\beta_1 + 0.264 \ge 8h(\gamma_1) + \varrho|\log\gamma_1| - \log|\gamma_1|.$$

Since $c = a + b + 2r < 4b$ we have $d > abc > c^2/4$ which implies $\beta_3 > \sqrt{cd} > \frac{1}{2}c^{3/2}$.

We can choose

$$a_2' := 28\log((1.264)^3\beta_3) > 60\log\left(1.264\cdot\frac{1}{\sqrt[3]{2}}\sqrt{c}\right) + 8\log((1.264)^3\beta_3) \ge$$
$$\ge \varrho|\log\gamma_2| - \log|\gamma_2| + 8h(\gamma_2).$$

From the assumption that $r > 10^4$ we have $a_1' > 56$ and $a_2' > 560$, so we see that our choice of parameters is valid and we can apply the theorem.

Table 1

$\gcd(s,8)$	1	2	4	8
h'	25.508	22.736	19.963	17.191
H	7.537	6.832	6.126	5.421
ω	4.005	4.006	4.007	4.0085
θ	1.07	1.076	1.085	1.097
C	0.02276	0.02284	0.02294	0.02307
C'	0.04696	0.04722	0.04753	0.04792

Set

$$b' := \frac{2s'}{a_2'} + 0.018 > \frac{b_1}{a_2'} + \frac{b_2}{a_1'}$$

and similarly as in the previous section

$$h' = 4\log b' + 12.6.$$

Since $\beta_3 = \frac{z+\sqrt{cd}}{2} < z$ and $z = st + 2 < s^3 + 2$ we have

$$h' > 4\log\left(\frac{s'}{14\log((1.264)^3(s^3+2))}\right) + 12.6.$$

Now for all four possible values of $\gcd(s,8)$ we calculate values from the Theorem 5 which are shown in Table 1.

Define also $B := \frac{1}{4}\left(h' + \frac{\lambda'}{\sigma}\right) < \log b' + 4.187$ which now yields

$$\begin{aligned}\log|\Lambda_2| &\geq -C\left(h' + \frac{\lambda'}{\sigma}\right)^2 a_1'a_2' - \sqrt{\omega\theta}\left(h' + \frac{\lambda'}{\sigma}\right) \\ &\qquad - \log\left(C'\left(h' + \frac{\lambda'}{\sigma}\right)^2 a_1'a_2'\right) \\ &\geq -C\cdot 16B^2a_1'a_2' - \sqrt{\omega\theta}\cdot 4B - \log(C'\cdot 16B^2a_1'a_2') \\ &\geq -0.3692B^2a_1'a_2' - 8.388B - \log(0.7668B^2a_1'a_2').\end{aligned}$$

On the other hand, from Lemma 23 we have

$$\log|\Lambda_2| < -4s'l_1\log\beta_1 + \log 2ad = -s'(a_1' - 0.264) + \log 2ad,$$

therefore

$$s'(a_1' - 0.264) < 0.3692B^2a_1'a_2' + 8.388B + \log(0.7668B^2a_1'a_2') + \log 2ad.$$

From $a_1' > 56$ we have $a_1' - 0.264 > 0.9952a_1'$, so now we can observe

$$\frac{2s'}{a'_2} < 0.74197B^2 + \frac{16.857}{a'_1a'_2}B + \frac{2.01}{a'_1a'_2}\log(0.7668B^2a'_1a'_2) + \frac{2.01}{a'_1a'_2}\log 2ad,$$

i.e.

$$b' < 0.74197B^2 + \frac{16.857}{a'_1a'_2}B + \frac{2.01}{a'_1a'_2}\log(0.7668B^2a'_1a'_2) + \frac{2.01}{a'_1a'_2}\log 2ad + 0.018.$$

Each addend on the right hand side of the inequality can be compared to B^2 and it leads to the inequality

$$b' < 0.742116B^2 + 0.02133 < 0.742116(\log b' + 4.187)^2 + 0.0213,$$

and from this we get $b' < 48.28$ which implies

$$s' < 24.131a'_2 < 675.668\log((1.264)^3(s^3+2)).$$

For each $\gcd(s,8) \in \{1,2,4,8\}$ we get that the upper bound for s is equal to $S_1 \in \{20610, 44324, 94814, 201884\}$ respectively, i.e. $s \le 201884$. □

Similarly we can prove the next lemma.

Lemma 25. *If* $8m \equiv 8n \equiv 0 \pmod{t}$, *then* $t \le 127293$.

Now we consider the last case from the Lemma 21.

Lemma 26. *If* $8n \equiv -4\varepsilon r \left(\bmod \frac{st}{\gcd(s,t)}\right)$, $\gcd(s,t) \in \{1,2,4\}$, *then* $r < 9164950$.

Proof. By Lemma 19 we see that $n > r/2$ which implies $8n + 4r > 8n - 4r > 0$, and depending on ε, we have $8n \pm 4r \ge \frac{st}{\gcd(s,t)} \ge \frac{st}{4}$. So, it always holds $n \ge \frac{st-16r}{32} \ge \frac{c(r-8)}{32}$. By Lemmas 14 and 12 we have $h > 2m > 2n$, which yields $h > \frac{c(r-8)}{16}$.

Moreover, from Proposition 6 we have

$$h < 3.46289 \cdot 10^{10} \log\alpha_2 \log c.$$

Since

$$\alpha_2 < \sqrt{ac+4} = \sqrt{\frac{16a}{r-8}\cdot\frac{c(r-8)}{16} + 4} < \sqrt{16\frac{c(r-8)}{16} + 4}$$

and

$$c = \frac{16}{r-8}\frac{c(r-8)}{16} < \frac{16}{10^{5/2}-8}\frac{c(r-8)}{16} < \frac{16}{308}\frac{c(r-8)}{16},$$

we have

$$\frac{c(r-8)}{16} < 3.46289 \cdot 10^{10} \log\left(\sqrt{16\frac{c(r-8)}{16}+4}\right) \log\left(\frac{16}{308}\frac{c(r-8)}{16}\right).$$

By direct calculation we get

$$\frac{c(r-8)}{16} < 1.57493 \cdot 10^{13}$$

and since $r^2 - 3 + 2r \geq c > 3r$ we have $r < 9164950$ and

$$h < 3.46289 \cdot 10^{10} \log(2r)\log(r^2-3+2r) < 1.85682 \cdot 10^{13}. \quad \square$$

From Lemmas 24, 25 and 26 we see that there are only finitely many triples $\{a, b, c\}$ left to check whether they are contained in a $D(4)$-quintuple. In order to deal with these remaining cases we will use a Baker–Davenport reduction method over a linear form

$$\Lambda_1 := 2h \log\frac{r+\sqrt{ab}}{2} - 2j \log\frac{s+\sqrt{ac}}{2} + \log\frac{\sqrt{c}(\sqrt{a}+\sqrt{b})}{\sqrt{b}(\sqrt{a}+\sqrt{c})}.$$

More explicitly, a modification of the Baker–Davenport reduction method, from [9], which we will use is stated next.

Lemma 27 *(Dujella, Pethő). Assume that M is a positive integer. Let p/q be the convergent of the continued fraction expansion of a real number κ such that $q > 6M$ and let*

$$\eta = \|\mu q\| - M \cdot \|\kappa q\|,$$

where $\|\cdot\|$ denotes the distance from the nearest integer. If $\eta > 0$, then the inequality

$$0 < J\kappa - K + \mu < AB^{-J}$$

has no solution in integers J and K with

$$\frac{\log(Aq/\eta)}{\log B} \leq J \leq M.$$

Consider the inequality $\frac{c(r-8)}{16} < 1.57493 \cdot 10^{13}$ from the proof of Lemma 26. For a fixed a we can calculate maximal r by putting $c = a + \frac{r^2-4}{a} + 2r$, and for smaller values of a we get a much better bound on r than the one calculated in the lemma. For example, for $a = 1$ we have $r \leq 63164$. Of course, we must also consider the bounds from Lemmas 24 and 25, where we have $a + r = s \leq 201884$ and $b + r = t \leq 127293$.

As we said before, we will apply Lemma 27 to the linear form in logarithms Λ_1, so we take $J = 2h$, $M = 2 \cdot 1.85682 \cdot 10^{13}$. It took approximately 29 hours and 45 minutes to run the algorithm in Wolfram Mathematica 11.1 package on the computer with Intel®

Core™ i7-4510U CPU @2.00-3.10 GHz processor and in each case we got $J = 2h < 5$ which cannot be true since $2h > 2 \cdot 0.666662\sqrt{ac} > 2 \cdot 0.666662 \cdot 10^{5/2} > 421$. This proves our next theorem.

Theorem 6. *A regular $D(4)$-triple $\{a, b, a+b+2r\}$ cannot be extended to a $D(4)$-quintuple.*

9. $D(4)$-quintuples with non-regular triples

It remains to show that a non-regular $D(4)$-triple cannot be extended to a quintuple. In the proof of the next two theorems we follow the methods used in Theorems 8 and 9 from [16], but as we also said before, results similar to those from [3], which we need in order to prove these Theorems, could not be proven for every $D(4)$-quintuple and here we will show how our results from Section 3 can again be used in proving some special results for $D(4)$-quintuples for which c is not minimal, i.e. $c \neq a + b + 2r$.

Theorem 7. *A $D(4)$-triple $\{a, b, c\}$ for which $\deg(a, b, c) = 1$ cannot be extended to a $D(4)$-quintuple.*

Proof. By Lemma 1 we have $c > \max\{ab, 4b\}$, and by Lemma 3 we also know $b > 4a$. Moreover, by the definition of the degree of a triple we know that $\{d_{-1}, a, b, c\}$ is a regular quadruple. Also, $\{d_{-1}, a, b\}$ is a regular triple, so if $d_{-1} > b$, we have $d_{-1} = a + b + 2r$, and if $d_{-1} < b$, it can be easily shown that $d_{-1} = a + b - 2r$. So, we have $d_{-1} = a + b \pm 2r$ and $c = d_+(a, d_{-1}, b) = r(r \pm a)(b \pm r)$.

For d_{-1} we have

$$d_{-1} \geq a + b - 2r \geq a + b - 2\sqrt{\frac{b^2}{4} + 4} > a - 1,$$

i.e. $d_{-1} \geq a$ so $c > abd_{-1} \geq a^2 b$.

We will now apply Lemma 7 to show that if $b \leq \max\{81a, 18.0793a^{3/2}\}$ then we cannot extend a triple $\{a, b, c\}$ of degree 1 to a $D(4)$-quintuple. For the remaining cases, where $b > \max\{81a, 18.0793a^{3/2}\}$, we will have more efficient bounds on the elements a and r and will be able to apply Baker–Davenport reduction.

First, we assume that $4a < b \leq 81a$. Since $c > a^2 b$ and $10^5 < b < 81a$, we have $a > 10^5/81$ and $c > 10^5/81 \cdot ab$ so we get

$$d > abc > \frac{10^5}{81} abab \geq \frac{10^5}{81} \frac{b^2}{81^2} b^2 = \frac{10^5}{81^3} b^4 > 18816 b^3.$$

On the other hand, since $b > 4a$ we have $b - a > 3a$ and $A' = \max\{4(B - A), 4A\} = 4(B - A)$, which yields

$$\frac{59.488A'B(B-A)^2}{Ag^4} = 237.952\frac{(b-a)^3b}{ag^4} < 237.952\frac{(b-a)^3b}{a}$$
$$\leq 237.952\left(\frac{80}{81}\right)^3 \cdot 81b^3 < 18570b^3,$$

so the conditions of the Lemma 7 are satisfied if we consider an extension of a $D(4)$-triple $\{a, b, d\}$ to a $D(4)$-quadruple. For the index n, (which refers to an extension to a quadruple and not a quintuple), we have by Lemma 7 that

$$n < \frac{4\log(8.40335 \cdot 10^{13}(A')^{\frac{1}{2}}A^{\frac{1}{2}}B^2Cg^{-1})\log(0.20533A^{\frac{1}{2}}B^{\frac{1}{2}}C(B-A)^{-1}g)}{\log(BC)\log(0.016858A(A')^{-1}B^{-1}(B-A)^{-2}Cg^4)}.$$

We can use $\frac{3}{4}b < b - a < \frac{80}{81}b$ and $1 \leq g = \gcd(a, b) \leq a$ and we observe expressions

$$8.40335 \cdot 10^{13}(A')^{\frac{1}{2}}A^{\frac{1}{2}}B^2Cg^{-1} < 8.35132 \cdot 10^{13}b^3d,$$
$$0.20533A^{\frac{1}{2}}B^{\frac{1}{2}}C(B-A)^{-1}g < 0.03423bd,$$
$$0.016858A(A')^{-1}B^{-1}(B-A)^{-2}Cg^4 > 0.0000544b^{-3}d,$$

thus we have

$$n < \frac{4\log(8.35132 \cdot 10^{13}b^3d)\log(0.03423bd)}{\log(bd)\log(0.000054b^{-3}d)}.$$

The function on the right hand side of the inequality is decreasing with d for $d > 0$, and since $d > 10^5 81^{-3}b^4 > 0.1881676b^4$ we obtain

$$n < \frac{4\log(1.571449 \cdot 10^{13}b^7)\log(0.006441b^5)}{\log(0.1881676b^5)\log(0.000010161b)}.$$

Similarly as in Proposition 1, we have

$$n \geq \frac{m}{2} > 0.309017\sqrt{ac} > 0.309017\sqrt{\frac{b}{81}\frac{10^5}{81}\frac{b}{81}b} > 0.134046b^{3/2}.$$

By combining these two inequalities we get $b < 98416 < 10^5$ which cannot be true. This means that our assumption was wrong and we have $b > 81a$.

Now we have an even better lower bound

$$d_{-1} > a + b - 2\sqrt{\frac{b^2}{81} + 4} > a + b - \frac{2}{9}\sqrt{b^2 + 324} > a + b - \frac{2}{9}(b+1) > \frac{7}{9}b$$

so

$$c > abd_{-1} > \frac{7}{9}ab^2$$

and $ac > \frac{7}{9}(ab)^2$.

Assume now that $81a < b \leq 18.0793a^{3/2}$. Obviously $a \geq 18.0793^{-2/3}b^{2/3}$. Observe that

$$\frac{59.488A'B(B-A)^2}{Ag^4} = 237.952\frac{(b-a)^3b}{ag^4} < 1639.12b^{10/3}.$$

On the other hand, since $d_{-1} > \frac{7}{9}b > \frac{7}{9}10^5$, we have

$$d > abc > d_{-1}a^2b^2 \geq d_{-1}18.0793^{-4/3}b^{10/3} > 1639.129b^{10/3},$$

so we can use Lemma 7 on triples $\{a, b, d\}$ where $81a < b < 18.0793a^{3/2}$. Notice that $A' = 4(B-A) < 4B$ and $1 \leq g \leq a < b/81$ so we use

$$8.40335 \cdot 10^{13}(A')^{\frac{1}{2}}A^{\frac{1}{2}}B^2Cg^{-1} < 1.86742 \cdot 10^{13}b^3d$$
$$0.20533A^{\frac{1}{2}}B^{\frac{1}{2}}C(B-A)^{-1}g < 0.0002852bd$$
$$0.016858A(A')^{-1}B^{-1}(B-A)^{-2}Cg^4 > 0.000611b^{-10/3}d,$$

to obtain

$$n < \frac{4\log(1.86742 \cdot 10^{13}b^3d)\log(0.0002852bd)}{\log(bd)\log(0.000611b^{-10/3}d)}.$$

Moreover

$$d > abc > d_{-1}a^2b^2 > \frac{7}{9}18.0793^{-4/3}b^{4/3}b^3 > 0.01639b^{13/3}$$

and since the function on the right hand side of inequality is decreasing with d, we can insert this lower bound on d and get

$$n < \frac{4\log(3.0608 \cdot 10^{11}b^{22/3})\log(4.6743 \cdot 10^{-6}b^{16/3})}{\log(0.01639b^{16/3})\log(1.001429 \cdot 10^{-5}b)}.$$

On the other hand,

$$n \geq \frac{m}{2} > 0.309017\sqrt{ac} > 0.272527ab > 0.272527 \cdot 18.0793^{-2/3}b^{2/3}b$$
$$> 0.03956b^{5/3}$$

which gives us $b \leq 99861$ after combining the inequalities. This, of course, leads to a contradiction which means that we must have $b > 18.0793a^{3/2}$.

From $b > 18.0793a^{3/2}$ we have

$$a^{5/2} < \frac{r^2-4}{18.0793}.$$

Since by Proposition 6, we have $ac < 1.17732 \cdot 10^{28}$, this implies $\frac{7}{9}(ab)^2 < 1.17732 \cdot 10^{28}$ i.e. $ab < 1.23033 \cdot 10^{14}$, which gives us $r \leq 11091997$ and $a \leq 135873$.

With these upper bounds, we again apply Baker–Davenport reduction on a linear form in logarithms Λ_1, with $J = 2h$, $M = 2 \cdot 7.23357 \cdot 10^{13}$. For each $\{a, b\}$ we check two options for c, namely $c = r(r \pm a)(b \pm r)$. It took 11 days and 18 hours to check all possibilities and in each case we had $J = 2h < 5$, which again cannot be true. This proves our theorem. □

All the remaining cases are covered in the next theorem which concludes the proof of Theorem 1.

Theorem 8. *A $D(4)$-triple $\{a, b, c\}$ such that $\deg(a, b, c) \geq 2$ cannot be extended to a $D(4)$-quintuple.*

Proof. From the assumption that $\deg(a, b, c) \geq 2$ we have that numbers $d_{-1} = d_-(a, b, c)$ and $d_{-2} = d_-(a, b, d_{-1})$ are positive integers. Moreover, here we also have $b > 4a$, $b > 10^5$ and $c > \max\{ab, 4b\}$.

Since from Proposition 1 we have an upper bound on c, we will separate our investigations into four subintervals

$$c \in \left\langle ab, a^{\frac{1}{2}} b^{\frac{3}{2}} \right] \cup \left\langle a^{\frac{1}{2}} b^{\frac{3}{2}}, ab^2 \right] \cup \left\langle ab^2, ab^{\frac{5}{2}} \right] \cup \left\langle ab^{\frac{5}{2}}, \frac{237.952 b^3}{a} \right].$$

Case I: $c \in \left\langle ab, a^{\frac{1}{2}} b^{\frac{3}{2}} \right]$.

Since $c = d_+(a, b, d_{-1})$, we have $c > abd_{-1}$ and $ad_{-1} < (ab)^{1/2}$, i.e. $ab > (ad_{-1})^2$. On the other hand, $ac > (ab)(ad_{-1}) > (ad_{-1})^3$, therefore

$$r_{(a,d_{-1})} = \sqrt{ad_{-1} + 4} < \sqrt{(1.17732 \cdot 10^{28})^{1/3} + 4} < 47697,$$

and since $d_{-1} \neq 0$, we also have $r_{(a,d_{-1})} \geq 3$. Our goal is to find for all $r \in [3, 47696]$ all possible pairs $\{a, d_{-1}\}$. Moreover, since $\{a, d_{-1}, b\}$ is a $D(4)$-triple, b is obtained as a solution of the generalized Pell equation

$$\mathcal{A}\mathcal{V}^2 - \mathcal{B}\mathcal{U}^2 = 4(\mathcal{A} - \mathcal{B})$$

where $\mathcal{A}\mathcal{B} + 4 = \mathcal{R}^2$, $\mathcal{A} < \mathcal{B}$ are positive integers. We know that all solutions of this equation are of the form

$$\mathcal{V}\sqrt{\mathcal{A}} + \mathcal{U}\sqrt{\mathcal{B}} = (\mathcal{V}_0\sqrt{\mathcal{A}} + \mathcal{U}_0\sqrt{\mathcal{B}}) \left(\frac{\mathcal{R} + \sqrt{\mathcal{A}\mathcal{B}}}{2} \right)^q,$$

where $q \geq 0$ is integer and $(\mathcal{U}_0, \mathcal{V}_0)$ is a solution which satisfies

$$0 \le \mathcal{U}_0 \le \sqrt{\frac{\mathcal{A}(\mathcal{B}-\mathcal{A})}{\mathcal{R}-2}}, \quad 1 \le |\mathcal{V}_0| \le \sqrt{\frac{(\mathcal{R}-2)(\mathcal{B}-\mathcal{A})}{\mathcal{A}}}.$$

Solutions can also be expressed as binary recurrence sequences

$$\mathcal{U}_0, \quad \mathcal{U}_1 = \frac{\mathcal{U}_0\mathcal{R}+\mathcal{V}_0\mathcal{A}}{2}, \quad \mathcal{U}_{m+2} = \mathcal{R}\mathcal{U}_{m+1} - \mathcal{U}_m.$$

Then we see that $b = (\mathcal{U}^2-4)/\mathcal{A} = (\mathcal{V}^2-4)/\mathcal{B}$, so it also must be true that $\mathcal{A}$ divides $\mathcal{U}^2-4$. Since $a^2b < ac < 1.17732\cdot 10^{28}$ we have $b < 1.17732\cdot 10^{28}a^{-2} \le 1.17732\cdot 10^{28}\mathcal{A}^{-2}$, so

$$\mathcal{U} < \sqrt{\frac{1.17732\cdot 10^{28}}{\mathcal{A}}+4}, \quad |\mathcal{V}| < \sqrt{\mathcal{B}\frac{1.17732\cdot 10^{28}}{\mathcal{A}^2}+4}.$$

Now, we describe an algorithm in which for each $\mathcal{R} = r_{(a,d_{-1})} \in [3, 47696]$ we search for divisors d' of $\mathcal{R}^2-4$ such that $1 \le d' \le \mathcal{R}$ and we set $\mathcal{A} = d'$ and $\mathcal{B} = (\mathcal{R}^2-4)/\mathcal{A}$. For a fixed pair $(\mathcal{A}, \mathcal{B})$ we find all possible solutions $(\mathcal{U}_0, \mathcal{V}_0)$ within the given bounds and for each pair we compute the quantities $\mathcal{U}_m$ until the upper bound for $\mathcal{U}$ is reached. For each $\mathcal{U}$ we check if $\mathcal{A}|\mathcal{U}^2-4$ and then take $b = (\mathcal{U}^2-4)/\mathcal{A}$ and for each possibility $(a, d_{-1}) \in \{(\mathcal{A}, \mathcal{B}), (\mathcal{B}, \mathcal{A})\}$ we can compute $c = d_+(a, b, d_{-1})$ and if $c \in \left\langle ab, a^{\frac{1}{2}}b^{\frac{3}{2}}\right]$ we can do Baker–Davenport reduction for the triple $\{a, b, c\}$ with parameters as in Theorem 2. It took 7 hours and 54 minutes to check all possibilities and we got $J < 5$ in each case.

Case II: $c \in \left\langle a^{\frac{1}{2}}b^{\frac{3}{2}}, ab^2\right]$.

We have $abd_{-1} < c < ab^2$, thus $d_{-1} < c/(ab) < b$, i.e. $b = \max\{a, b, d_{-1}\}$. By Lemma 2 we have

$$a^{\frac{1}{2}}b^{\frac{3}{2}} < c < ad_{-1}b + 4b = b(ad_{-1}+4),$$

which yields

$$(ab)^{1/2} < ad_{-1} + 4.$$

Similarly, $d_{-2} = d_-(a, b, d_{-1})$, therefore $b > ad_{-1}d_{-2}$ and $d_{-2} < b/(ad_{-1}) < b/((ab)^{1/2}-4)$. Now we have

$$ad_{-2} < (ab)^{1/2}\frac{(ab)^{1/2}}{(ab)^{1/2}-4} = (ab)^{1/2}\left(1+\frac{4}{(ab)^{1/2}-4}\right) < 1.01282(ab)^{1/2},$$

and also we can see that $ab > ((ad_{-2})/1.01282)^2$. Moreover

$$ad_{-2} < 1.01282(ad_{-1}+4) = 1.01282ad_{-1} + 4.05128,$$

so

$$\frac{ad_{-2} - 4.05128}{1.01282} < ad_{-1}.$$

Now,

$$ac > (ab)(ad_{-1}) > \left(\frac{ad_{-2}}{1.01282}\right)^2 \frac{ad_{-2} - 4.05128}{1.01282}$$

and since $ac < 1.17732 \cdot 10^{28}$, we get $ad_{-2} < 2.30408 \cdot 10^9$ and

$$r_{(a,d_{-2})} = \sqrt{ad_{-2} + 4} < 48001.$$

We also know that $d_{-1} < b < c^{2/3} < (1.17732 \cdot 10^{28})^{2/3} < 35.17524 \cdot 10^{18}$. Similarly as in the first case we apply an algorithm in which for each $\mathcal{R} = r_{(a,d_{-2})}$ we search for pairs $(\mathcal{A}, \mathcal{B})$. We set $d_{-1} = (\mathcal{U}^2 - 4)/\mathcal{A}$ and observe both possibilities $(a, d_{-2}) \in \{(\mathcal{A}, \mathcal{B}), (\mathcal{B}, \mathcal{A})\}$ and define $b = d_+(a, d_{-1}, d_{-2})$, $c = d_+(a, b, d_{-1})$. It took 1 hour and 34 minutes to do the reduction and we got $J < 5$ in each case.

Case III: $c \in \left\langle ab^2, a^{\frac{3}{2}} b^{\frac{5}{2}}\right]$.

Here we have $(ab)^2 < ac < 1.17732 \cdot 10^{28}$, so $r = \sqrt{ab+4} \le 10416543$. It can be shown that $b < d_{-1} < c/(ab)$, therefore we have $d_{-1} < a^{3/2}b^{5/2}/(ab) = a^{1/2}b^{3/2}$. Since $b < d_{-1}$, we have $d_{-1} = d_+(a, b, d_{-2})$ and $d_{-1} > abd_{-2}$, i.e.

$$ad_{-2} < \frac{d_{-1}}{b} < (ab)^{1/2} < r \le 10416543$$

and $r_{(a,d_{-2})} = \sqrt{ad_{-2} + 4} < 3228$.

The algorithm is similar as in Case II., except b and d_{-1} exchange roles, so $b = \frac{\mathcal{U}^2 - 1}{\mathcal{A}}$, and $d_{-1} = d_+(a, b, d_{-2})$. It took less than 3 minutes to check all possibilities and we got $J < 5$ in each case.

Case IV: $c \in \left\langle a^{\frac{3}{2}} b^{\frac{5}{2}}, \frac{237.952b^3}{a}\right]$.

Here we have $a^{3/2}b^{5/2} < 237.952b^3/a$, which yields $b > a^5/237.952^2$ and

$$1.17732 \cdot 10^{28} > ac > (ab)^{5/2} > (a^6 \cdot 237.952^{-2})^{5/2},$$

therefore we get $a \le 460$. As in Case III., we have $b < d_{-1}$, $d_{-1} = d_+(a, b, d_{-2})$ and $c = d_+(a, b, d_{-1})$. Therefore

$$d_{-1} < \frac{c}{ab} < \frac{237.952b^2}{a^2}, \quad d_{-2} < \frac{d_{-1}}{ab} < \frac{237.952b}{a^3}.$$

From $(ab)^{5/2} < 1.17732 \cdot 10^{28}$ we have $ab < 1.69184 \cdot 10^{11}$. Also, from $a^4 d_{-2} < 237.952ab$ we get $ad_{-2} < 4.02576 \cdot 10^{13}/a^3$, thus $r_{(a,d_{-2})} < 6344883/a^{3/2}$.

Since $a \le 460$, it is more efficient if we search $r_{(a,d_{-2})}$ inside interval $\left[3, \frac{6344883}{a^{3/2}}\right]$ such that $a | r^2_{(a,d_{-2})} - 4$ for each fixed a. We set $d_{-2} = (r^2_{(a,d_{-2})} - 4)/a$ and do similarly as in the previous cases. It took 9 days and 21 hour to check all possibilities and again we got $J < 5$ in each case. $\square$

Acknowledgments

The authors are supported by Croatian Science Foundation under the project no. 6422. The authors are also very grateful to Mihai Cipu who made many important remarks and suggestions on previous version of this paper.

References

[1] Y.M. Aleksentsev, The Hilbert polynomial and linear forms in the logarithms of algebraic numbers, Izv. Math. 72 (2008) 1063–1110.
[2] M. Bliznac, A. Filipin, An upper bound for the number of Diophantine quintuples, Bull. Aust. Math. Soc. 94 (3) (2016) 384–394.
[3] M. Cipu, A. Filipin, Y. Fujita, Bounds for Diophantine quintuples II, Publ. Math. Debrecen 86 (2016) 59–78.
[4] M. Cipu, Y. Fujita, Bounds for Diophantine quintuples, Glas. Mat. Ser. III 50 (2015) 25–34.
[5] M. Cipu, T. Trudgian, Searching for Diophantine quintuples, Acta Arith. 173 (2016) 365–382.
[6] A. Dujella, Diophantine m-tuples, http://web.math.pmf.unizg.hr/~duje/dtuples.html.
[7] A. Dujella, There are only finitely many Diophantine quintuples, J. Reine Angew. Math. 566 (2004) 183–214.
[8] A. Dujella, M. Mikić, On the torsion group of elliptic curves induced by $D(4)$-triples, An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat. 22 (2) (2014) 79–90.
[9] A. Dujella, A. Pethő, A generalization of a theorem of Baker and Davenport, Quart. J. Math. Oxford Ser. (2) 49 (1998) 291–306.
[10] A. Dujella, A.M.S. Ramasamy, Fibonacci numbers and sets with the property $D(4)$, Bull. Belg. Math. Soc. Simon Stevin 12 (3) (2005) 401–412.
[11] A. Filipin, There does not exist a $D(4)$-sextuple, J. Number Theory 128 (2008) 1555–1565.
[12] A. Filipin, On the size of sets in which $xy + 4$ is always a square, Rocky Mountain J. Math. 39 (4) (2009) 1195–1224.
[13] A. Filipin, An irregular $D(4)$-quadruple cannot be extended to a quintuple, Acta Arith. 136 (2) (2009) 167–176.
[14] A. Filipin, There are only finitely many $D(4)$-quintuples, Rocky Mountain J. Math. 41 (6) (2011) 1847–1859.
[15] A. Filipin, The extension of some $D(4)$-pairs, Notes Number Theory Discrete Math. 23 (2017) 126–135.
[16] B. He, A. Togbé, V. Ziegler, There is no Diophantine quintuple, Trans. Amer. Math. Soc. (2018), https://doi.org/10.1090/tran/7573, in press.
[17] M. Laurent, Linear forms in two logarithms and interpolation determinants II, Acta Arith. 133 (4) (2008) 325–348.
[18] E.M. Matveev, An explicit lower bound for a homogeneous rational linear form in logarithms of algebraic numbers II, Izv. Math. 64 (2000) 1217–1269.
[19] M. Mignotte, A kit on linear forms in three logarithms, preprint, available at http://www-irma.u-strasbg.fr/~bugeaud/travaux/kit.pdf.