Total positivity of Toeplitz matrices of recursive Hypersequences

Došlić, Tomislav; Martinjak, Ivica; Škrekovski, Riste

Source / Izvornik: **Ars Mathematica Contemporanea, 2019, 17(1), 125 - 139**

Journal article, Published version
Rad u časopisu, Objavljena verzija rada (izdavačev PDF)

https://doi.org/10.26493/1855-3974.1526.b8d

ARS MATHEMATICA CONTEMPORANEA

ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 17 (2019) 125–139
https://doi.org/10.26493/1855-3974.1526.b8d
(Also available at http://amc-journal.eu)

Total positivity of Toeplitz matrices of recursive hypersequences*

Tomislav Došlić
Faculty of Civil Engineering, University of Zagreb, Zagreb, Croatia

Ivica Martinjak
Faculty of Science, University of Zagreb, Zagreb, Croatia

Riste Škrekovski
FMF, University of Ljubljana, Ljubljana, Slovenia
Faculty of Information Studies, Novo Mesto, Slovenia
FAMNIT, University of Primorska, Koper, Slovenia

Received 9 November 2017, accepted 12 March 2019, published online 6 September 2019

Abstract

We present a new class of totally positive Toeplitz matrices composed of recently introduced hyperfibonacci numbers of the r-th generation. As a consequence, we obtain that all sequences $F_n^{(r)}$ of hyperfibonacci numbers of r-th generation are log-concave for $r \geq 1$ and large enough n.

Keywords: Total positivity, totally positive matrix, Toeplitz matrix, Hankel matrix, hyperfibonacci sequence, log-concavity.

Math. Subj. Class.: 15B36, 15A45

1 Introduction and preliminary results

A matrix M is *totally positive* if all its minors are positive real numbers. When it is allowed that minors are zero, then M is said to be *totally non-negative*. Such matrices appears in many areas having numerous applications including, among other topics, graph theory,

*Partial support of the Croatian Science Foundation through project BioAmpMode (Grant no. 8481) is gratefully acknowledged by the first author. All authors gratefully acknowledge partial support of Croatian-Slovenian bilateral project “Modeling adsorption on nanostructures: A graph-theoretical approach” BI-HR/16-17-046. The third author is partially supported by Slovenian research agency ARRS, program no. P1-0383.

E-mail addresses: doslic@grad.hr (Tomislav Došlić), imartinjak@phy.hr (Ivica Martinjak), skrekovski@gmail.com (Riste Škrekovski)

Pólya frequency sequences, oscillatory motion, symmetric functions and quantum groups among these areas [1, 2, 12, 13, 18]. The notion of total positivity is closely related with log-concavity and more on this one can find in a paper by Stanley [21]. A classical result by Whitney, Loewer and Cryer [8] says that any totally non-negative matrix M can be factored as a product of totally non-negative matrices $M = L_1 \cdots L_m D U_1 \cdots U_m$, where D is a diagonal matrix with non-negative elements, L_i is a matrix of the form $I + cE_{j+1,j}$, U_i is a matrix of the form $I + cE_{j,j+1}$ and $E_{k,l}$ is the matrix which has a 1 on the k, l position and zeros elsewhere. There is also a connection between totally non-negative matrices and planar networks proved by Karlin and McGregor [15], and Lindström [16]. The famous Lindström lemma gives combinatorial interpretation of a minor through the weights of collections of vertex-disjoint paths in a planar network.

An important notion when testing a matrix on total positivity is *initial minor*. We let I, J denote column set and row set, respectively. A minor $\Delta_{I,J}$ where both I and J consist of several consecutive indices and where $I \cup J$ contain 1, is called *initial*. Thus, each matrix entry is the lower-right corner of exactly one initial minor. In this work we use Theorem 1.1, which is proved by Gasca and Peña [14].

Theorem 1.1. *A square matrix is totally positive if and only if all its initial minors are positive.*

The notion of total positivity can be refined as follows. A matrix M is said to be *totally positive of order* p (or TP_p, in short) if all its minors of all orders $\le p$ are positive.

The concept of total positivity extends in a straightforward manner also to (semi)infinite matrices. It turns out that many such triangular matrices appearing in combinatorics are indeed TP [3]. Recently, Wang and Wang proved total positivity of Catalan triangle via Aissen-Schonberg-Whitney theorem [22]. Further general results on triangular matrices and Riordan array have been obtained by Chen, Liang and Wang [5, 6] as well as Zhao and Yan [23], while Pan and Zeng give combinatorial interpretation of results on total positivity of Catalan-Stieltjes matrices [20].

A *Toeplitz* matrix $T = [t_{i,j}]$ is a (finite or infinite) matrix whose entries satisfy $t_{i,j} = t_{i+1,j+1}$. In finite case,

$$T = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-n+1} \\ t_1 & t_0 & \cdots & t_{-n+2} \\ \vdots & \vdots & \ddots & \vdots \\ t_{n-1} & t_{n-2} & \cdots & t_0 \end{pmatrix}.$$

In words, elements of a Toeplitz matrix are constant along diagonals descending from left to right. If the elements of a matrix are constant along diagonals ascending from left to right, the matrix is called a *Hankel matrix*. An example is given here,

$$H = \begin{pmatrix} t_0 & t_1 & \cdots & t_{n-1} \\ t_1 & t_2 & \cdots & t_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ t_{n-1} & t_{n-2} & \cdots & t_{2n-2} \end{pmatrix}.$$

Obviously, each Toeplitz (or Hankel) matrix of order n gives rise to a unique sequence (of length $2n - 1$ in the finite case) of its elements. The connection also works the other way:

Given an (infinite) sequence (a_n) and given integers n_0 and m, we can construct a Toeplitz (or a Hankel) matrix of order m having a_{n_0} in the upper left corner. In what follows we present a class of totally positive Toeplitz matrices whose entries are hyperfibonacci numbers [4, 17, 24]. These sequences of numbers were recently introduced by Dil and Mező in a study of a symmetric algorithm for hyperharmonic and some other integer sequences [9].

Definition 1.2. The hyperfibonacci sequence of the r-th generation $(F_n^{(r)})_{n\geq 0}$ is a sequence arising from the recurrence relation

$$F_n^{(r)} = \sum_{k=0}^{n} F_k^{(r-1)}, \quad F_n^{(0)} = F_n, \quad F_0^{(r)} = 0, \quad F_1^{(r)} = 1, \tag{1.1}$$

where $r \in \mathbb{N}$ and F_n is the n-th term of the Fibonacci sequence, $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0, F_1 = 1$.

Proposition 1.3 gives some basic identities for hyperfibonacci sequences [7].

Proposition 1.3. *For hyperfibonacci sequence $(F_n^{(r)})_{n\geq 0}$ we have*

(i)

$$F_n^{(r)} = F_{n-1}^{(r)} + F_n^{(r-1)} \tag{1.2}$$

(ii)

$${F_n^{(1)}}^2 - F_{n-1}^{(1)} F_{n+1}^{(1)} = F_{n-3}^{(1)} + 1 + (-1)^{n+1}$$

(iii)

$$F_n^{(1)} F_{n+1}^{(1)} - F_{n-1}^{(1)} F_{n+2}^{(1)} = F_{n-2}^{(1)} + 1 - (-1)^{n+1}$$

(iv)

$$F_n^{(r)} = F_{n+2r} - \sum_{k=0}^{r-1} \binom{n+r+k}{r-1-k}. \tag{1.3}$$

Explicit formula for determinant of the Hankel matrix of hyperfibonacci sequence of r-th generation

$$A_{r,n} = \begin{pmatrix} F_n^{(r)} & F_{n+1}^{(r)} & \cdots & F_{n+r+1}^{(r)} \\ F_{n+1}^{(r)} & F_{n+2}^{(r)} & \cdots & F_{n+r+2}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ F_{n+r+1}^{(r)} & F_{n+r+2}^{(r)} & \cdots & F_{n+2r+2}^{(r)} \end{pmatrix}$$

has been obtained in [19] and here we state it in Theorem 1.4. We will find it useful in establishing our main result, the total positivity of the Toeplitz matrix of the same sequence with odd-indexed hyperfibonacci number in the upper left corner.

Theorem 1.4. *For the sequence $(F_k^{(r)})_{k\geq 0}$, $r \in \mathbb{N}$ and $n \in \mathbb{N}$ a determinant of a matrix $A_{r,n}$ takes values ± 1,*

$$\det(A_{r,n}) = (-1)^{n+\left\lfloor \frac{r+3}{2} \right\rfloor}.$$

The TP_2 property of Toeplitz and Hankel matrices is closely related to log-concavity and log-convexity, respectively, of the associated sequences. Recall that a sequence (a_n) of positive numbers is *log-concave* if $a_n^2 \geq a_{n-1}a_{n+1}$ holds for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. If the inequality is reversed, the sequence is *log-convex*. The literature on log-concavity and log-convexity is vast. Besides already mentioned classical papers by Stanley [21] and Brenti [3], we refer the reader also to [10, 11, 20, 22] for some recently developed techniques. In particular, the log-concavity of hyperfibonacci numbers of all generations $r \geq 1$ has been established in [24] by using recurrence relations. Here we proceed to prove more general claims that will imply the log-concavity results of reference [24].

2 Positivity of hyperfibonacci determinant

We let $B_{m,n}^{(r)} = [b_{i,j}]$ denote the matrix of order m consisting of hyperfibonacci numbers of the r-th generation,

$$B_{m,n}^{(r)} := \begin{pmatrix} F_n^{(r)} & F_{n-1}^{(r)} & \cdots & F_{n-m+1}^{(r)} \\ F_{n+1}^{(r)} & F_n^{(r)} & \cdots & F_{n-m+2}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ F_{n+m-1}^{(r)} & F_{n+m-2}^{(r)} & \cdots & F_n^{(r)} \end{pmatrix}$$

with the constraint $r \geq m-1$. In what follows we will show that there exist $q(r) \in \mathbb{N}$ such that $\det(B_{m,n}^{(r)})$ is positive for $n \geq q(r)$.

From the elementary properties of the Fibonacci sequence known as Cassini identity we immediately have that the matrix

$$M = \begin{pmatrix} F_{2n+1} & F_{2n+2} \\ F_{2n+2} & F_{2n+3} \end{pmatrix}$$

is positive for $n \in \mathbb{N}_0$ and the matrix

$$M' = \begin{pmatrix} F_{2n+1} & F_{2n} \\ F_{2n+2} & F_{2n+1} \end{pmatrix}$$

is positive for $n \in \mathbb{N}$. In Proposition 2.1 we extend the property of positivity to matrices of order 2 consisting from first generation of hyperfibonacci numbers while a general result, involving r-th generation of hyperfibonacci numbers is given in Theorem 3.5.

Proposition 2.1. *For $n, r \in \mathbb{N}$ determinant of the matrix $B_{2,n}^{(1)}$ is positive,*

$$\det(B_{2,n}^{(1)}) = \det \begin{pmatrix} F_n^{(1)} & F_{n-1}^{(1)} \\ F_{n+1}^{(1)} & F_n^{(1)} \end{pmatrix} > 0.$$

Proof. We apply relations presented in Proposition 1.3 to get $F_n^{(1)} - F_{n-1}^{(1)} = F_n$. Now, by the properties of determinant (column subtraction and then row subtraction) we obtain

$$\det \begin{pmatrix} F_n^{(1)} & F_{n-1}^{(1)} \\ F_{n+1}^{(1)} & F_n^{(1)} \end{pmatrix} = \det \begin{pmatrix} F_n & F_{n-1}^{(1)} \\ F_{n+1} & F_n^{(1)} \end{pmatrix} = \det \begin{pmatrix} F_n & F_{n-1}^{(1)} \\ F_{n-1} & F_n \end{pmatrix}$$
$$= \det \begin{pmatrix} F_n & F_{n+1} - 1 \\ F_{n-1} & F_n \end{pmatrix} > 0. \qquad \square$$

Theorem 2.2. *Let* $m \in \mathbb{N}$. *Then there is* $n_m \in \mathbb{N}$ *such that* $\det\left(B_{m,n}^{(m-1)}\right) > 0$ *for all* $n \geq n_m$.

Proof. Employing elementary transformation on matrices and using relation (1.2) we get

$$\det\left(B_{m,n}^{(m-1)}\right) = \det\begin{pmatrix} F_n & F_{n-1}^{(1)} & F_{n-2}^{(2)} & \cdots & F_{n-m+1}^{(m-1)} \\ F_{n+1} & F_n^{(1)} & F_{n-1}^{(2)} & \cdots & F_{n-m+2}^{(m-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ F_{n+m-1} & F_{n+m-2}^{(1)} & F_{n+m-3}^{(2)} & \cdots & F_n^{(m-1)} \end{pmatrix}$$

$$= \det\begin{pmatrix} F_n & F_{n-1}^{(1)} & F_{n-2}^{(2)} & \cdots & F_{n-m+1}^{(m-1)} \\ F_{n-1} & F_n & F_{n-1}^{(1)} & \cdots & F_{n-m+2}^{(m-2)} \\ \vdots & \vdots & \vdots & & \vdots \\ F_{n-m+2} & F_{n-m+3} & F_{n-m+4} & \cdots & F_{n-1}^{(1)} \\ F_{n-m+1} & F_{n-m+2} & F_{n-m+3} & \cdots & F_n \end{pmatrix}. \tag{2.1}$$

Having in mind relation (1.3) we immediately obtain

$$F_{n-r}^{(r)} = F_{n+r} - \sum_{k=0}^{r-1} \binom{n+k}{r-1-k}$$

and furthermore

$$F_{n-r}^{(r)} = F_{n+r} - S_r, \tag{2.2}$$

where

$$S_r := \sum_{k=0}^{r-1} \binom{n+k}{r-1-k}.$$

Thus, $S_1 = 1$, $S_2 = n+1$, $S_3 = \frac{n(n-1)}{2} + n + 2$, $S_4 = \frac{n^3+5n}{6} + n + 3$, etc. Now, we substitute entries in (2.1) according to (2.2) to get

$$\det\left(B_{m,n}^{(m-1)}\right) = \det\begin{pmatrix} F_n & F_{n+1} - S_1 & F_{n+2} - S_2 & \cdots & F_{n+m-1} - S_{m-1} \\ F_{n-1} & F_n & F_{n+1} - S_1 & \cdots & F_{n+m-2} - S_{m-2} \\ \vdots & \vdots & \vdots & & \vdots \\ F_{n-m+1} & F_{n-m+2} & F_{n-m+3} & \cdots & F_n \end{pmatrix}. \tag{2.3}$$

In the following steps of this proof we let Δ_1, Δ_2, Δ_3 denote matrices we deal with. We will show that determinants of these matrices are equal to each other. In order to make the proof more readable, the elements of the last two columns of Δ_1, Δ_2, Δ_3 are denoted by $c_{i,j}$, $c'_{i,j}$, $c''_{i,j}$, respectively. On the other hand, the elements of the first $m-2$ columns of these matrices are denoted by $b_{i,j}$ and they do not change their values under performed transformation.

When performing elementary transformations on matrix columns of (2.3) we obtain

$$\det\left(B_{m,n}^{(m-1)}\right) = \det\begin{pmatrix} S_2-S_1 & S_3-S_2-S_1 & \cdots & F_{n+m-2}-S_{m-2} & F_{n+m-1}-S_{m-1} \\ S_1 & S_2-S_1 & \cdots & F_{n+m-3}-S_{m-3} & F_{n+m-2}-S_{m-2} \\ 0 & S_1 & \cdots & F_{n+m-4}-S_{m-4} & F_{n+m-3}-S_{m-3} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & F_n & F_{n+1}-S_1 \\ 0 & 0 & \cdots & F_{n-1} & F_n \end{pmatrix}$$
$$= \det(\Delta_1)$$

where we get $\Delta_1 = [b_{i,j}]$ by similar transformation on rows,

$$\Delta_1 = \begin{pmatrix} S_2-2S_1 & S_3-2S_2-S_1 & \cdots & -S_{m-1}+S_{m-2}+S_{m-3} \\ S_1 & S_2-2S_1 & \cdots & -S_{m-2}+S_{m-3}+S_{m-4} \\ 0 & S_1 & \cdots & -S_{m-3}+S_{m-4}+S_{m-5} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & S_2-S_1 \\ 0 & 0 & \cdots & F_{n+1}-S_1 \\ 0 & 0 & \cdots & F_n \end{pmatrix},$$

$$\begin{aligned} b_{i,j} &= b_{i+1,j+1}, \quad i=1,\dots,m-1, \quad j=1,\dots,m-3, \\ b_{i,j} &= c_{i,j}, \quad i=1,\dots,m, \quad j=m-1,m, \\ c_{i,m-1} &= c_{i+1,m}, \quad i=1,\dots,m-3 \end{aligned}$$

and where entries $b_{i,j}$ get values

$$\begin{aligned} b_{1,1} &= S_2-2S_1 \\ b_{1,2} &= S_3-2S_2-S_1 \\ b_{1,3} &= S_4-2S_3-S_2+2S_1 \\ b_{1,4} &= S_5-2S_4-S_3+2S_2+S_1 \\ b_{1,5} &= S_6-2S_5-S_4+2S_3+S_2 \\ &\vdots \\ b_{1,m-2} &= S_{m-1}-2S_{m-2}-S_{m-3}+2S_{m-4}+S_{m-5}, \end{aligned}$$

while for entries $c_{i,j}$ we have

$$\begin{aligned} c_{1,m-1} &= -S_{m-2}+S_{m-3}+S_{m-4} \\ c_{2,m-1} &= -S_{m-3}+S_{m-4}+S_{m-5} \\ &\vdots \\ c_{m-3,m-1} &= -S_2+S_1 \\ c_{m-2,m-1} &= -S_1 \\ c_{m-1,m-1} &= F_n \\ c_{m,m-1} &= F_{n+1}, \end{aligned}$$

and

$$\begin{aligned} c_{m-1,m} &= F_{n+1} - S_1 \\ c_{m,m} &= F_n. \end{aligned}$$

Furthermore, we form matrix $\Delta_2 = [b_{i,j}]$ with $b_{i,j} = c'_{i,j},\ \ i = 1, \ldots, m,\ \ j = m-1, m,$ by performing row transformations

$$\begin{aligned} c'_{i,m-1} &= c_{i,m-1} + \sum_{j=1}^{m-3} b_{i,j},\ \ i = 1, \ldots, m \\ c'_{i,m} &= c_{i,m} + \sum_{j=1}^{m-2} b_{i,j},\ \ i = 1, \ldots, m. \end{aligned}$$

As a consequence of these two operations for the last two columns of Δ_2 we obtain

$$\begin{pmatrix} -S_{m-4} + S_{m-6} + S_{m-7} + \cdots + S_2 & -S_{m-3} + S_{m-5} + S_{m-6} + \cdots + S_2 \\ \vdots & \vdots \\ -S_4 + S_2 & -S_5 + S_3 + S_2 \\ -S_3 & -S_4 + S_2 \\ -S_2 & -S_3 \\ -S_1 & -S_2 \\ 0 & -S_1 \\ 0 & 0 \\ F_n & F_{n+1} \\ F_{n-1} & F_n \end{pmatrix}.$$

(while the other entries of Δ_2 are equal to those of Δ_1). Clearly, $\det(\Delta_1) = \det(\Delta_2)$. Furthermore, we perform row transformations

$$\begin{aligned} c''_{i,m-1} &= c'_{i,m-1} + b_{i,m-5} + 2b_{i,m-6} + 4b_{i,m-7} + \cdots + (F_{m-3} - 1)b_{i,1} \\ c''_{i,m} &= c'_{i,m-1} + b_{i,m-4} + 2b_{i,m-5} + 4b_{i,m-6} + \cdots + (F_{m-2} - 1)b_{i,1} \end{aligned}$$

to get matrix $\Delta_3 = [b_{i,j}]$ where $b_{i,j} = c''_{i,j},\ \ i = 1, \ldots, m,\ \ j = m-1, m$. Then, the last two columns of Δ_3 are

$$\begin{pmatrix} -F_{m-2} & -F_{m-1} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ F_n & F_{n+1} \\ F_{n-1} & F_n \end{pmatrix}.$$

Namely, a straighforward but tedious algebraic manipulation give us a nice value for $c''_{1,m-1}$,

$$\begin{aligned} c''_{1,m-1} &= (F_{m-6} - 1)S_1 + (F_{m-5} - 1)2S_1 - (F_{m-4} - 1)S_1 - (F_{m-3} - 1)2S_1 \\ &= -F_{m-2}. \end{aligned}$$

In the same fashion one can prove that $c''_{1,m} = -F_{m-1}$ and $c''_{i,j} = 0$, $i = 2, \ldots, m-2$, $j = m-1, m$. Again, determinant is not affected under these transformations, $\det(\Delta_3) = \det(\Delta_2)$.

We shall now separately treat the matrix Δ_3, for even and odd n. Using the Fibonacci recurrence relation, for even n we immediately obtain

$$\det\left(B_{m,n}^{(m-1)}\right) = \det\begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,m-2} & -F_{m-2} & -F_{m-1} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,m-2} & 0 & 0 \\ 0 & b_{3,2} & \cdots & b_{3,m-2} & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & b_{m-2,m-2} & 0 & 0 \\ 0 & 0 & \cdots & F_{n-1} & 0 & 1 \\ 0 & 0 & \cdots & -F_{n-2} & 1 & 1 \end{pmatrix}$$

$$= -\det\begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b'_{1,m-2} & -F_{m-3} & -F_{m-2} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,m-2} & 0 & 0 \\ 0 & b_{3,2} & \cdots & b_{3,m-2} & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & b_{m-2,m-2} & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

where $b'_{1,n-2} = b_{1,m-2} + F_{m-3}F_{n-1} - F_{m-2}F_{n-2}$. This determinant can be represented as the sum of the upper triangular determinants. Now we use the fact that there is $q \in \mathbb{N}$ such that the Fibonacci number F_q is bigger that the value $P(q)$, $F_q > P(q)$, where $P(n)$ is a polynomial of any degree. The only element in the matrix above containing Fibonacci numbers is $b'_{1,m-2}$. The fact that the term $F_{n-1}F_{m-3}$ has a positive contribution in the determinant completes the proof for case when n is even.

When n is odd we have

$$\det\left(B_{m,n}^{(m-1)}\right) = \det\begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,m-2} & -F_{m-2} & -F_{m-1} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,m-2} & 0 & 0 \\ 0 & b_{3,2} & \cdots & b_{3,m-2} & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & b_{m-2,m-2} & 0 & 0 \\ 0 & 0 & \cdots & F_{n-2} & 1 & 1 \\ 0 & 0 & \cdots & -F_{n-1} & 0 & 1 \end{pmatrix}.$$

Now, analogue arguments as when n is even completes the proof. □

In particular, when $m = 4$ we have

$$\det\left(B_{4,n}^{(3)}\right) = \det\begin{pmatrix} S_2 - 2 & S_3 - 2S_2 - 1 & -1 & -2 \\ 1 & S_2 - 2 & 0 & 0 \\ 0 & 1 & F_n & F_{n+1} \\ 0 & 0 & F_{n-1} & F_n \end{pmatrix}.$$

When n is even then

$$\det\left(B_{4,n}^{(3)}\right) = \det\begin{pmatrix} S_2-2 & S_3-2S_2-1 & -1 & -2 \\ 1 & S_2-2 & 0 & 0 \\ 0 & F_{n-1} & 0 & 1 \\ 0 & -F_{n-2} & 1 & 1 \end{pmatrix}$$

$$= \det\begin{pmatrix} S_2-2 & S_3-2S_2-1-F_{n-2}+F_{n-1} & -1 & -1 \\ 1 & S_2-2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= -(S_2-2)\begin{pmatrix} S_2-2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} S_3-2S_2-1-F_{n-2}+F_{n-1} & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= -(n-1)^2 + \frac{n(n-1)}{2} - n - 1 + F_{n-3}.$$

The inequality

$$F_{n-3} > (n-1)^2 - \frac{n(n-1)}{2} + n + 1$$

holds true for $n \geq 15$ and consequently $\det\left(B_{4,n}^{(3)}\right) > 0$ for $n \geq 15$ when n is even. Similarly, when n is odd

$$\det\left(B_{4,n}^{(3)}\right) = (S_2-2)\begin{pmatrix} S_2-2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} S_3-2S_2-1-F_{n-3} & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= (n-1)^2 - \frac{n(n-1)}{2} + n + 1 + F_{n-3}.$$

Thus, it follows from these two cases that $\det\left(B_{4,n}^{(3)}\right) > 0$ for $n \geq 15$.

Note that the proof of Theorem 2.2 can be used to efficient calculation of determinants of matrices $B_{m,n}^{(m-1)}$. We will illustrate this on the example for $m = 4$ and $n = 5$. In that case, when applying the proof of Theorem 2.2 we have

$$\det\left(B_{4,5}^{(3)}\right) = \det\begin{pmatrix} 51 & 25 & 11 & 4 \\ 97 & 51 & 25 & 11 \\ 176 & 97 & 51 & 25 \\ 309 & 176 & 97 & 51 \end{pmatrix} = \det\begin{pmatrix} 6 & 11 & -1 & -1 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= 24 - 11 = 13.$$

Corollary 2.3. *Let $m, n, r \in \mathbb{N}$ and $r \geq m-1$. Then there is $q \in \mathbb{N}$ such that determinant of the matrix $B_{m,n}^{(r)}$ is positive for all $n \geq q$,*

$$\det\left(B_{m,n}^{(r)}\right) > 0.$$

Proof. We proceed by induction on r. The base case, $r = m - 1$, is provided by Theorem 2.2. Let us now assume that the claim is true for $m - 1 \leq p \leq r - 1$. Our task is to show that the determinant

$$\det(B_{m,n}^{(r)}) = \det \begin{pmatrix} F_n^{(r)} & F_{n-1}^{(r)} & \cdots & F_{n-m+1}^{(r)} \\ F_{n+1}^{(r)} & F_n^{(r)} & \cdots & F_{n-m+2}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ F_{n+m-1}^{(r)} & F_{n+m-2}^{(r)} & \cdots & F_n^{(r)} \end{pmatrix}$$

is also positive. We first recall (1.2) and then start subtracting rows of $B_{m,n}^{(r)}$. We subtract $(m-1)$-st row from m-th, then $(m-2)$-nd from $(m-1)$-st, and continue all the way down till we subtract the first row from the second. Since the determinant remains unchanged, we obtain

$$\det(B_{m,n}^{(r)}) = \det \begin{pmatrix} F_n^{(r)} & F_{n-1}^{(r)} & \cdots & F_{n-m+1}^{(r)} \\ F_{n+1}^{(r-1)} & F_n^{(r-1)} & \cdots & F_{n-m+2}^{(r-1)} \\ \vdots & \vdots & \ddots & \vdots \\ F_{n+m-1}^{(r-1)} & F_{n+m-2}^{(r-1)} & \cdots & F_n^{(r-1)} \end{pmatrix}.$$

We expand the determinant on the right hand side over the elements of the first row.

$$\begin{aligned} \det(B_{m,n}^{(r)}) &= F_n^{(r)}\Delta_1 + \cdots + F_{n-m+1}^{(r)}\Delta_m \\ &= \frac{F_n^{(r)}}{F_n^{(r-1)}} F_n^{(r-1)}\Delta_1 + \cdots + \frac{F_{n-m+1}^{(r)}}{F_{n-m+1}^{(r-1)}} F_{n-m+1}^{(r-1)}\Delta_m, \end{aligned}$$

where Δ_i denotes the determinant obtained from $\det(B_{m,n}^{(r)})$ by omitting the first row and i-th column for $1 \leq i \leq m$. Let us denote $x_i = \frac{F_{n-i+1}^{(r)}}{F_{n-i+1}^{(r-1)}}$ and define a function $f\colon \mathbb{R}^m \to \mathbb{R}$ by

$$f(x_1, \ldots, x_m) = \sum_{i=0}^{m} x_{i+1} F_{n-i}^{(r-1)} \Delta_i.$$

Obviously, $f(1, \ldots, 1) = \det(B_{m,n}^{(r-1)}) > 0$, and hence $f(c, \ldots, c) = c \cdot \det(B_{m,n}^{(r-1)}) > 0$, for any positive constant c. In particular, $f(\phi^2, \ldots, \phi^2) > 0$, where $\phi^2 = \frac{3+\sqrt{5}}{2}$.

Since f is continuous, there must exist a neighborhood

$$W = (\phi^2 - \delta_1, \phi^2 + \delta_1) \times \cdots \times (\phi^2 - \delta_m, \phi^2 + \delta_m)$$

such that f is positive on W. Now we use the explicit expression

$$F_n^{(r)} = F_{n+2r} - \sum_{k=0}^{r-1} \binom{n+r+k}{r-1-k}$$

from Proposition 1.3. By dividing it through by analogous expression for $F_n^{(r-1)}$ and passing to limit when $n \to \infty$, one readily obtains

$$\lim_{n\to\infty} \frac{F_n^{(r)}}{F_n^{(r-1)}} = \phi^2.$$

That further implies that, for large enough n, the coefficient $x_i = \frac{F^{(r)}_{n-i+1}}{F^{(r-1)}_{n-i+1}}$ falls into $(\phi^2 - \delta_i, \phi^2 + \delta_i)$ for all i, and hence

$$f\left(\frac{F^{(r)}_n}{F^{(r-1)}_n}, \dots, \frac{F^{(r)}_{n-m+1}}{F^{(r-1)}_{n-m+1}}\right) = \det(B^{(r)}_{m,n}) > 0.$$

That completes the proof. □

3 Main results

We let $T_{r,n}$ denote the matrix of order $r+2$ consisting of hyperfibonacci numbers of the r-th generation,

$$T_{r,n} := \begin{pmatrix} F^{(r)}_{2n+1} & F^{(r)}_{2n} & \cdots & F^{(r)}_{2n-r} \\ F^{(r)}_{2n+2} & F^{(r)}_{2n+1} & \cdots & F^{(r)}_{2n-r+1} \\ \vdots & \vdots & \ddots & \vdots \\ F^{(r)}_{2n+r+2} & F^{(r)}_{2n+r+1} & \cdots & F^{(r)}_{2n+1} \end{pmatrix}.$$

Lemma 3.1. *For $n \in \mathbb{N}$ and the hyperfibonacci sequence $\left(F^{(1)}_n\right)_{n\geq 0}$ the matrix*

$$T_{1,n} = \begin{pmatrix} F^{(1)}_{2n+1} & F^{(1)}_{2n} & F^{(1)}_{2n-1} \\ F^{(1)}_{2n+2} & F^{(1)}_{2n+1} & F^{(1)}_{2n} \\ F^{(1)}_{2n+3} & F^{(1)}_{2n+2} & F^{(1)}_{2n+1} \end{pmatrix}$$

is totally positive.

Proof. According to Proposition 2.1 the three initial minors of order 2 of $T_{1,n}$ are positive. It is immediately seen from Theorem 1.4 that determinant $\det(T_{1,n})$ is positive. These facts complete the proof. □

Note that the matrix $T_{1,n} = [t_{i,j}]$ is a Toeplitz matrix, with the element $t_{1,1}$ being hyperfibonacci number of the first generation having odd index. If we allow both even and odd indices for $t_{1,1}$ then the property of total positivity is lost. Such determinant of order 3 in not positive for even indices (by Theorem 1.4), while it keeps the positivity of minors of order 2. We express this fact, that follows from the proof of Lemma 3.1, in Corollary 3.2.

Corollary 3.2. *For $n \in \mathbb{N}$ and the hyperfibonacci sequence $\left(F^{(1)}_n\right)_{n\geq 0}$ the matrix*

$$T'_{1,n} = \begin{pmatrix} F^{(1)}_{n} & F^{(1)}_{n-1} & F^{(1)}_{n-2} \\ F^{(1)}_{n+1} & F^{(1)}_{n} & F^{(1)}_{n-1} \\ F^{(1)}_{n+2} & F^{(1)}_{n+1} & F^{(1)}_{n} \end{pmatrix}$$

is TP_2.

Lemma 3.3. *For $n \geq 4$ and the hyperfibonacci sequence $\left(F^{(2)}_n\right)_{n\geq 0}$ the matrix*

$$T_{2,n} = \begin{pmatrix} F^{(2)}_{2n+1} & F^{(2)}_{2n} & F^{(2)}_{2n-1} & F^{(2)}_{2n-2} \\ F^{(2)}_{2n+2} & F^{(2)}_{2n+1} & F^{(2)}_{2n} & F^{(2)}_{2n-1} \\ F^{(2)}_{2n+3} & F^{(2)}_{2n+2} & F^{(2)}_{2n+1} & F^{(2)}_{2n} \\ F^{(2)}_{2n+4} & F^{(2)}_{2n+3} & F^{(2)}_{2n+2} & F^{(2)}_{2n+1} \end{pmatrix}$$

is totally positive.

Proof. According to Proposition 2.1 the five initial minors of order 2 of $T_{2,n}$ are positive. Furthermore, the three initial minors of order 3 are positive when $n \geq 3$ by Corollary 2.3. However, when $n = 3$ determinant $\det(T_{2,n})$ is negative (by Theorem 1.4) so the matrix $T_{2,n}$ is totally positive for $n \geq 4$. □

Having in mind Proposition 2.1 and the fact that the matrix $B^{(2)}_{3,n}$ has positive determinant for $n \geq 7$ we immediately derive Corollary 3.4.

Corollary 3.4. *For $n \geq 8$ and the hyperfibonacci sequence $\left(F^{(2)}_n\right)_{n\geq 0}$ the matrix*

$$T'_{2,n} = \begin{pmatrix} F^{(2)}_n & F^{(2)}_{n-1} & F^{(2)}_{n-2} & F^{(2)}_{n-3} \\ F^{(2)}_{n+1} & F^{(2)}_n & F^{(2)}_{n-1} & F^{(2)}_{n-2} \\ F^{(2)}_{n+2} & F^{(2)}_{2n+1} & F^{(2)}_n & F^{(2)}_{n-1} \\ F^{(2)}_{n+3} & F^{(2)}_{2n+2} & F^{(2)}_{2n+1} & F^{(2)}_n \end{pmatrix}$$

is TP$_3$.

Furthermore, it holds true that

$$\det(B^{(3)}_{4,n}) > 0, \quad n \geq 15$$
$$\det(B^{(4)}_{4,n}) > 0, \quad n \geq 5.$$

When $r \geq 5$ there is no constraint on the value of n when asking for positivity of $\det(B^{(r)}_{4,n})$.

Theorem 3.5. *For the hyperfibonacci sequence $\left(F^{(r)}_n\right)_{n\geq 0}$ there is $q \in \mathbb{N}$ such that the matrix $T_{r,n}$ of order $r+2$*

$$T_{r,n} = \begin{pmatrix} F^{(r)}_{2n+1} & F^{(r)}_{2n} & \cdots & F^{(r)}_{2n-r} \\ F^{(r)}_{2n+2} & F^{(r)}_{2n+1} & \cdots & F^{(r)}_{n-r+1} \\ \vdots & \vdots & \ddots & \vdots \\ F^{(r)}_{n+r+2} & F^{(r)}_{2n+r+1} & \cdots & F^{(r)}_{2n+1} \end{pmatrix}$$

is totally positive for $n \geq q$.

Proof. First we prove that $2n+1$ initial minors of order 2 are positive. These submatrices are of the form $B^{(r)}_{2,m_2}$ where $m_2 > 2n - r$, so there they have positive determinant for $r \geq 1$ and $n \geq 1$, according to Corollary 2.3. Obviously, another initial minors are of the form

$$B^{(r)}_{3,m_3}, B^{(r)}_{4,m_4}, \ldots, B^{(r)}_{r+1,m_{r+1}}.$$

According to Corollary 2.3 there exist numbers $q_3, q_4, \ldots, q_{r+1} \in \mathbb{N}$ such that

$$\det(B^{(r)}_{3,m_3}) > 0, \quad m_3 \geq q_3$$
$$\det(B^{(r)}_{4,m_4}) > 0, \quad m_4 \geq q_4$$
$$\vdots$$
$$\det(B^{(r)}_{r+1,m_{r+1}}) > 0, \quad m_{r+1} \geq q_{r+1}.$$

It remains to show that $\det(T_{r,n})$ is itself positive. We start by noticing that $T_{r,n}$ can be obtained from $A_{r,2n-r}$ by reversing the order of columns. That corresponds to right multiplication of $A_{r,2n-r}$ by U_{r+2}, where U_{r+2} is a square matrix of order $r+2$ whose elements are $(U_{r+2})_{i,j} = 1$ if $i+j = r+3$ and zero otherwise. It is immediately seen that $\det(U_{r+2}) = (-1)^{\lfloor (r+2)/2 \rfloor}$. Now we have $\det(T_{r,n}) = \det(A_{r,2n-r})\det(U_{r+2})$, and Theorem 1.4 implies

$$\det(T_{r,n}) = (-1)^{2n-r+\lfloor (r+3)/2 \rfloor + \lfloor (r+2)/2 \rfloor} = (-1)^2 = 1,$$

for all r. That completes the proof. $\square$

We conclude the section with another result that follows directly from Corollary 3.4.

Corollary 3.6. *For the hyperfibonacci sequence $\left(F_n^{(r)}\right)_{n\geq 0}$ there is $q \in \mathbb{N}$ such that the matrix $T'_{r,n}$ of order $r+2$*

$$T'_{r,n} = \begin{pmatrix} F_n^{(r)} & F_{n-1}^{(r)} & \cdots & F_{n-r-1}^{(r)} \\ F_{n+1}^{(r)} & F_n^{(r)} & \cdots & F_{n-r}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ F_{n+r+1}^{(r)} & F_{n+r}^{(r)} & \cdots & F_n^{(r)} \end{pmatrix}$$

is TP$_{r+1}$ for $n \geq q$.

4 Concluding remarks

In this paper we have considered several classes of Toeplitz matrices associated to sequences of hyperfibonacci numbers of given generation. We have established various positivity results for such matrices. In particular, we showed that such matrices with odd-indexed hyperfibonacci numbers on the main diagonal are totally positive for large enough values of index n. When the restriction to odd-valued indices is omitted, the total positivity is not preserved, but we established that those matrices are TP$_{r+1}$ for a given generation r and large enough n. That implies (at least asymptotical) log-concavity of hyperfibonacci numbers of all generations $r \geq 1$. Our results thus extend and strengthen results of reference [24] established by a different approach. It would be interesting to have combinatorial proofs of log-concavity of $F_n^{(r)}$ for $r \geq 1$; at the moment, we are not aware of any.

We have also tried to explore the form of dependence of q_r on r. The numerical evidence, collected in Table 1, suggests that $2q_r+1$, the index in the upper left corner, behaves as $7r-5$ for even r and $7r-4$ for r odd. It would be interesting to examine whether the

Table 1: Some values of parameter q_r in Theorem 3.5.

r	1	2	3	4	5	6	7	8	9	10	11
$2q_r+1$	5	9	17	23	31	37	45	51	59	65	73

pattern (or at least a linear dependence) persists for larger r, and if it does, to find some explanation.

We are fairly confident that the methods and results presented here could be extended so as to encompass also other sequences defined by two-term recurrences and their iterated partial sums. It would be worthwhile to explore whether the same approach could be applicable to the sequences defined by longer linear recurrences with constant coefficients, such as the sequence of tribonacci numbers.

References

[1] T. Ando, Totally positive matrices, *Linear Algebra Appl.* **90** (1987), 165–219, doi:10.1016/0024-3795(87)90313-2.

[2] A. Berenstein, S. Fomin and A. Zelevinsky, Parametrizations of canonical bases and totally positive matrices, *Adv. Math.* **122** (1996), 49–149, doi:10.1006/aima.1996.0057.

[3] F. Brenti, Combinatorics and total positivity, *J. Comb. Theory Ser. A* **71** (1995), 175–218, doi:10.1016/0097-3165(95)90000-4.

[4] N.-N. Cao and F.-Z. Zhao, Some properties of hyperfibonacci and hyperlucas numbers, *J. Integer Seq.* **13** (2010), Article 10.8.8 (11 pages), https://cs.uwaterloo.ca/journals/JIS/VOL13/Cao2/cao5r.html.

[5] X. Chen, H. Liang and Y. Wang, Total positivity of recursive matrices, *Linear Algebra Appl.* **471** (2015), 383–393, doi:10.1016/j.laa.2015.01.009.

[6] X. Chen, H. Liang and Y. Wang, Total positivity of Riordan arrays, *European J. Combin.* **46** (2015), 68–74, doi:10.1016/j.ejc.2014.11.009.

[7] L. Cristea, I. Martinjak and I. Urbiha, Hyperfibonacci sequences and polytopic numbers, *J. Integer Seq.* **19** (2016), Article 16.7.6 (13 pages), https://cs.uwaterloo.ca/journals/JIS/VOL19/Urbiha/urbiha4.html.

[8] C. W. Cryer, Some properties of totally positive matrices, *Linear Algebra Appl.* **15** (1976), 1–25, doi:10.1016/0024-3795(76)90076-8.

[9] A. Dil and I. Mező, A symmetric algorithm for hyperharmonic and Fibonacci numbers, *Appl. Math. Comput.* **206** (2008), 942–951, doi:10.1016/j.amc.2008.10.013.

[10] T. Došlić, Log-balanced combinatorial sequences, *Int. J. Math. Math. Sci.* (2005), 507–522, doi:10.1155/ijmms.2005.507.

[11] T. Došlić and D. Veljan, Logarithmic behavior of some combinatorial sequences, *Discrete Math.* **308** (2008), 2182–2212, doi:10.1016/j.disc.2007.04.066.

[12] S. Fomin and A. Zelevinsky, Total positivity: tests and parametrizations, *Math. Intelligencer* **22** (2000), 23–33, doi:10.1007/bf03024444.

[13] M. Gasca and C. A. Micchelli (eds.), *Total Positivity and Its Applications*, volume 359 of *Mathematics and its Applications*, Kluwer Academic Publishers Group, Dordrecht, 1996, doi:10.1007/978-94-015-8674-0, proceedings of the meeting held in Jaca, September 26 – 30, 1994.

[14] M. Gasca and J. M. Peña, Total positivity and Neville elimination, *Linear Algebra Appl.* **165** (1992), 25–44, doi:10.1016/0024-3795(92)90226-z.

[15] S. Karlin and G. McGregor, Coincidence probabilities, *Pacific J. Math.* **9** (1959), 1141–1164, doi:10.2140/pjm.1959.9.1141.

[16] B. Lindström, On the vector representations of induced matroids, *Bull. London Math. Soc.* **5** (1973), 85–90, doi:10.1112/blms/5.1.85.

[17] R. Liu and F.-z. Zhao, On the sums of reciprocal hyperfibonacci numbers and hyperlucas numbers, *J. Integer Seq.* **15** (2012), Article 12.4.5 (10 pages), https://cs.uwaterloo.ca/journals/JIS/VOL15/Liu/liu10.html.

[18] G. Lusztig, Introduction to total positivity, in: J. Hilgert, J. D. Lawson, K.-H. Neeb and E. B. Vinberg (eds.), *Positivity in Lie Theory: Open Problems*, De Gruyter, Berlin, volume 26 of *De Gruyter Expositions in Mathematics*, pp. 133–145, 1998, doi:10.1515/9783110811186.133.

[19] I. Martinjak and I. Urbiha, A new generalized Cassini determinant, *Colloq. Math.* **145** (2016), 209–218, doi:10.4064/cm6756-11-2015.

[20] Q. Pan and J. Zeng, On total positivity of Catalan-Stieltjes matrices, *Electron. J. Combin.* **23** (2016), #P4.33 (18 pages), https://www.combinatorics.org/ojs/index.php/eljc/article/view/v23i4p33.

[21] R. Stanley, Log-concave and unimodal sequences in algebra, combinatorics, and geometry, in: M. F. Capobianco, M. G. Guan, D. F. Hsu and F. Tian (eds.), *Graph Theory and Its Applications: East and West*, New York Academy of Sciences, New York, volume 576 of *Annals of the New York Academy of Sciences*, 1989 pp. 500–535, doi:10.1111/j.1749-6632.1989.tb16434.x, proceedings of the First China-USA International Conference held in Jinan, June 9 – 20, 1986.

[22] C. Z.-C. Wang and Y. Wang, Total positivity of Catalan triangle, *Discrete Math.* **338** (2015), 566–568, doi:10.1016/j.disc.2014.11.017.

[23] L. Zhao and F. Yan, Note on total positivity for a class of recursive matrices, *J. Integer Seq.* **19** (2016), Article 16.6.5 (6 pages), https://cs.uwaterloo.ca/journals/JIS/VOL19/Zhao/zhao17.html.

[24] L.-N. Zheng, R. Liu and F.-Z. Zhao, On the log-concavity of the hyperfibonacci numbers and the hyperlucas numbers, *J. Integer Seq.* **17** (2014), Article 14.1.4 (9 pages), https://cs.uwaterloo.ca/journals/JIS/VOL17/Zheng/zheng5.html.