Quadratic Chabauty for Atkin-Lehner quotients of modular curves of prime level and genus 4, 5, 6

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by

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1. Introduction. The curve $X_0^+(N)$ is the quotient of the modular curve $X_0(N)$ by the Atkin–Lehner involution w_N (also called the Fricke involution). The non-cuspidal points of $X_0^+(N)$ classify unordered pairs of elliptic curves together with a cyclic isogeny of degree N between them, where the Atkin–Lehner involution w_N sends an isogeny to its dual. The set $X_0^+(N)(\mathbb{Q})$ consists of cusps, points corresponding to CM elliptic curves (CM points), and points corresponding to quadratic $\mathbb{Q}$-curves without complex multiplication. The points of the third kind are referred to as *exceptional points*.

There have been several works related to the study of $\mathbb{Q}$-rational points on Atkin–Lehner quotients of modular curves (see [BDM$^+$21, CB09, DLF21, EL23, Gal96, Gal99, Gal02, Mer18, Mom87]) and on Atkin–Lehner quotients of Shimura curves (see [Cla03, PY07]). Especially relevant for this work are the articles [Gal96, Gal99, Gal02] in which Galbraith constructs models of all such curves of genus ≤ 5 except for $X_0^+(263)$, and he conjectures that he has found all exceptional points on these curves. Building on work of Galbraith, Mercuri [Mer18] constructs models for such curves of genus 6 and 7 of prime level and shows that up to a (very) large naive height, there are no exceptional points on five of these curves (those with $N = 163, 197, 229, 269$ and 359).

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Towards verifying Galbraith's conjecture, Balakrishnan, Dogra, Müller, Tuitman, and Vonk [BDM+21] show that of the curves $X_0^+(N)$ of prime level N and genus 2 and 3, the only curves with exceptional rational points are those with level $N = 73, 103, 191$. The rational points on $X_0^+(N)$ for $N = \{67, 73, 103\}$ were computed by Balakrishnan, Best, Bianchi, Lawrence, Müller, Triantafillou, and Vonk [BBB+21] using quadratic Chabauty and the Mordell–Weil sieve. The remaining genus 2 levels, $\{107, 167, 191\}$, are more challenging because there are not enough rational points to run the quadratic Chabauty algorithm [BBB+20]. The authors of [BDM+21] compute the rational points on the remaining curves of genus 2 and prime level, as well as those of genus 3 and prime level, using models computed by Galbraith [Gal96] and Elkies.

Quadratic Chabauty refers to the technique of depth-2 Chabauty–Kim. The Chabauty–Kim method [Kim05, Kim09] can be seen as a generalization of the usual Chabauty–Coleman method. The usual Chabauty–Coleman method is a useful tool in computing rational points on curves (see [Cha41, Col85, MP12, Sto19]), however there still are large classes of curves that do not satisfy the rank condition $r := \operatorname{rk}\operatorname{Jac}(X)(\mathbb{Q}) < g$ of Chabauty–Coleman. Kim developed a non-abelian version of the Chabauty–Coleman method in which one replaces the use of the Jacobian of a curve with a Selmer variety associated to a certain Galois-stable unipotent quotient of the $\mathbb{Q}_p$-pro-unipotent completion of the étale fundamental group of the curve. Despite the technical nature of Kim's setup, the work of Balakrishnan–Dogra and Balakrishnan, Dogra, Müller, Tuitman, and Vonk [BD18, BD21, BD19, BDM+19] has shown that one can practically implement a depth-two version of Kim's program. In particular, work of Siksek [Sik17] uses the criterion of Balakrishnan–Dogra for finiteness of the quadratic Chabauty set to show that for modular curves of genus $g \geq 3$, quadratic Chabauty is more likely to succeed than classical Chabauty–Coleman. Further, Balakrishnan, Best, Bianchi, Dogra, Lawrence, Müller, Triantafillou, Tuitman, and Vonk developed computational tools to carry out quadratic Chabauty explicitly (see [BM20, BDM+21, BBB+20]).

Continuing the work begun in [BDM+21], we aim to compute all $\mathbb{Q}$-rational points on the curves $X_0^+(N)$ of prime level N when the genus is $4, 5, 6$ using the quadratic Chabauty method [BD18, BD21] and the computational tools developed in [BBB+20]. From Section 4, we see that for prime level N, the curve $X_0^+(N)$ has genus 4 if and only if

$$N \in \{137, 173, 199, 251, 311\}. \tag{1.1}$$

It has genus 5 if and only if

$$N \in \{157, 181, 227, 263\}, \tag{1.2}$$

and it has genus 6 if and only if

$$N \in \{163, 197, 211, 223, 269, 271, 359\}. \tag{1.3}$$

These modular curves are not hyperelliptic according to [HH96]. They satisfy the conditions required for quadratic Chabauty because their Mordell–Weil rank is equal to their genus (1) and their Picard number is greater than 1 since $J_0^+(N)$ has real multiplication. In addition, they usually have enough rational points (according to [BDM+19, Remark 1.6], $g+1$ points suffice when using equivariant p-adic heights, and one can lower this requirement by constructing g independent rational points on the Jacobian of the curve).

To apply the quadratic Chabauty method, we will compute suitable plane affine patches of the curves $X_0^+(N)$. As a starting point for this, we use the canonical models for $X_0^+(N)$ computed by Galbraith [Gal96, Gal99, Gal02] and Mercuri [Mer18]. The equations for $N \in \{137, 157, 173, 181, 199, 227, 251\}$ are taken (2) from [Gal96, pp. 32–34], the equation for $N = 311$ is taken from [Gal99, pp. 316], and the equations for $N \in \{163, 197, 211, 223, 269, 271, 359\}$ are taken from [Mer18, pp. 299–306]. The rational points are taken from [Gal96, pp. 88], [Gal99, pp. 316], and [Mer18, pp. 299–306]. For $N \in \{199, 251\}$, we computed the rational points by brute force. These equations and the list of known rational points in each of these levels are given in Section 5.

The main difficulty of applying the `QCMod` code developed in [BBB+20] to these canonical models lies in finding suitable plane affine patches (in the sense of the last paragraph of Section 2) as inputs for the `QCModAffine` function in `QCMod`. Realizing this step enables us to verify that the list of rational points is complete. We explain our method of finding the affine patches in Section 3, and list them in Section 5. It was surprising that we could go up to genus 6 with the current techniques.

Our main result is the following:

THEOREM 1.1. *For prime level N, the only curves $X_0^+(N)$ of genus* 4 *that have exceptional rational points are $X_0^+(137)$ and $X_0^+(311)$. For prime level N, there are no exceptional rational points on curves $X_0^+(N)$ of genus* 5 *and* 6*.*

Therefore, this work confirms Galbraith's conjecture for prime levels. In particular, this, combined with the results of [BDM+21] on genus 2 and 3 curves as well as with [AM10, BBB+21, BDM+19, BGX21, Mom86, Mom87,

(1) In all our examples, every element f of the newform Galois orbits of the modular abelian variety $J_0^+(N)$ satisfies $\mathrm{ord}_{s=1} L(f,s) = 1$. So the rank equals the genus by Gross–Zagier–Kolyvagin–Logachëv [GZ86, KL89]. In particular, the hypotheses of the classical Chabauty–Coleman method are never satisfied.

(2) The canonical model for $X_0^+(157)$ given in Section 5 corrects a small typo in the third equation in Galbraith's model for $N = 157$ (the leading term should be $2w^2$ instead of w^2).

AM22], resolves Galbraith's conjecture for both prime and composite levels. For genus 6, this work shows that the subset of rational points that Mercuri found is in fact all of them, and for those curves with $N = 163, 197, 359$ we confirm [Mer18, Result 1.2] without any restrictions on height ([3]).

Our paper is organized as follows. In Section 2, we give an overview of the quadratic Chabauty method. Section 3 contains a description of the method we applied, that is, how we found the models that we used to run the quadratic Chabauty algorithm starting from the canonical model of $X_0^+(N)$, and how we chose suitable primes. In Section 4, we bound the genus of $X_0^+(N)$ from below and give a list of all levels N such that the genus of $X_0^+(N)$ is less than or equal to 6. In Section 5, we give tables for genus $4, 5, 6$ containing the canonical models of $X_0^+(N)$, the known rational points, and our models and primes for which we run quadratic Chabauty.

The code supporting the claims made in this paper was written in Magma [BCP97] and may be found at [AAB$^+$21].

2. Overview of quadratic Chabauty for modular curves: theory and algorithm. Let $X/\mathbb{Q}$ be a smooth projective geometrically connected curve of genus $g \geq 2$ with Jacobian J whose Mordell–Weil group $J(\mathbb{Q})$ has rank $r = g$. For a prime p of good reduction, the abelian logarithm induces a homomorphism

$$\log\colon J(\mathbb{Q}_p) \to \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^\vee.$$

We assume that the p-adic closure of the image of $J(\mathbb{Q}) \subset J(\mathbb{Q}_p)$ in $\mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^\vee$ has rank g (otherwise classical Chabauty–Coleman applies). We also assume that $X(\mathbb{Q})$ is non-empty, so we can choose a rational base point b to map X into its Jacobian J. If the Néron–Severi rank ρ of J is larger than 1, then there exists a non-trivial $Z \in \mathrm{Ker}(\mathrm{NS}(J) \to \mathrm{NS}(X))$ inducing a correspondence on $X \times_{\mathbb{Q}} X$. Balakrishnan and Dogra [BD18, BDM$^+$19] explain how to attach to any such Z a locally analytic *quadratic Chabauty function*

$$\rho_Z\colon X(\mathbb{Q}_p) \to \mathbb{Q}_p$$

as follows: using Nekovář's theory of p-adic heights [Nek93], one can construct a global p-adic height which decomposes as a sum of local height functions. The quadratic Chabauty function ρ_Z is defined as the difference between the global p-adic height and the local height for the chosen prime p. Even though we do not go into details here, we note that the computation of the

([3]) For composite levels, one can further consider the quotient $X_0^*(N)$ by all Atkin–Lehner involutions; in this case, the $\mathbb{Q}$-points have been determined in [ACKP22] if the quotient is hyperelliptic.

global height pairing is easier when X has many rational points (at least three, which is indeed the case for all modular curves studied in this paper).

The crucial property of the quadratic Chabauty function is that there exists a finite set $\Upsilon \subset \mathbb{Q}_p$ such that $\rho_Z(x) \in \Upsilon$ for any $x \in X(\mathbb{Q})$. Because ρ_Z has Zariski-dense image on every residue disk and is given by a convergent power series, this implies that the set $X(\mathbb{Q}_p)_2$ of $\mathbb{Q}_p$-points of X having values in Υ under ρ_Z is finite. Because $X(\mathbb{Q})$ is contained in that set, it is finite as well. Since both ρ_Z and Υ can be explicitly computed by [BBB+20], this makes the provable determination of $X(\mathbb{Q}_p)_2$ possible ([4]) if the quadratic Chabauty condition $r < g + \rho - 1$ is satisfied (e.g., if $r = g$ and $\rho > 1$).

In [BDM+21], the authors applied quadratic Chabauty to some of the modular curves associated to congruence subgroups of $\mathrm{SL}_2(\mathbf{Z})$, as well as Atkin–Lehner quotients of such curves. The algorithm in [BDM+21] is specific to modular curves only when determining the non-trivial class Z. In that algorithm, Z is computed by the Hecke operator T_p (which is determined by the Eichler–Shimura relation). We recall the input and output of the algorithm in [BDM+21] here:

- Input:
  - A plane affine patch $Y\colon Q(x,y) = 0$ of a modular curve $X/\mathbb{Q}$ that satisfies $r = g \geq 2$ and is such that the p-adic closure of the image of $J(\mathbb{Q})$ in $\mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)^\vee$ under log has rank g.
  - A prime p which is a prime of good reduction for $X/\mathbb{Q}$ such that the Hecke operator T_p generates $\mathrm{End}(J) \otimes_{\mathbb{Z}} \mathbb{Q}$ as a $\mathbb{Q}$-algebra. We check this condition in our cases by showing that $a_p(f)$ generates the number field generated by the coefficients of f where f is any newform orbit representative associated to $X/\mathbb{Q}$.
- Output: A finite set containing $X(\mathbb{Q}_p)_2$.

The curves of our interest, $X_0^+(N)$ of genus 4, 5 and 6 and prime level N, satisfy the condition $r = g$. This is seen by checking that for all N in (1.1), (1.2) and (1.3), $f \in S_2(\Gamma_0(N))^{+,\mathrm{new}}$ satisfies $L'(f,1) \neq 0$ (actually, we just need to check this for one arbitrary representative in each Galois orbit). By [KL89], this implies that the (algebraic and analytic) rank of $J_0^+(N)/\mathbb{Q}$ equals $\sum_i \dim(A_{f_i}) = g$ where the summation is taken over an arbitrary set of newform orbit representatives. However, it is not necessarily the case that $r = g$ for all $X_0^+(N)$ of prime level N. The smallest genus g for which there exists a prime N such that $r > g$ is $g = 206$, with $N = 5077$ [BDM+21, DLF21].

([4]) At least under the assumption that ρ_Z has no repeated roots; otherwise, we get a finite superset.

The implementation (available at [BBB$^+$20]) of the quadratic Chabauty algorithm in [BDM$^+$21] is designed to take as input a plane affine patch $Y\colon Q(x,y)=0$ of a modular curve $X/\mathbb{Q}$. The model $Q(x,y)=0$ does not need to be smooth, but it should be monic in y with p-integral coefficients. We can sometimes find an affine patch Y such that all rational points on X must be among the points returned by running their program on Y. We have managed to do so for all genus 4 and 5 curves $X_0^+(N)$ of prime level N. If no such Y is found, then we find two affine patches such that every rational point on X is contained in at least one patch. Moreover, we need every $\mathbb{F}_p$-point on X (realized as a canonical model) to map to a good point (Definition 3.1) on at least one patch. Finding such favorable affine patches turned out to be the most challenging part when applying their algorithm for large genus Atkin–Lehner quotients. We talk about our strategy for generating these affine patches in Section 3.

3. Overview of strategy. As introduced in Section 2, our first task is to find a suitable plane model $Q(x,y)$ of $X_0^+(N)$ that is monic in y and has "small" coefficients. We start with the image of the canonical embedding of $X_0^+(N)$ in $\mathbf{P}^{g-1}$ (as stated in the introduction, these are non-hyperelliptic for the levels N we consider). In what follows we shall identify $X_0^+(N)$ with its canonical image and use the same notation for both. We find two rational maps $\tau_x,\tau_y\colon X_0^+(N)\to\mathbf{P}^1$ such that the product $\tau_x\times\tau_y\colon X_0^+(N)\to\mathbf{P}^1\times\mathbf{P}^1$ is a birational map onto its image ([5]). In practice, we used Magma's `Genus4GonalMap` to find such maps, and analogous functions for genus 5 and 6. Compose with the Segre embedding $\mathbf{P}^1\times\mathbf{P}^1\to\mathbf{P}^3_{[w:x:y:z]}$, and then project from the point $[1:0:0:0]$ onto the plane $w=0$; denote by φ' the composite

$$\varphi'\colon X_0^+(N)\xrightarrow{\tau_x\times\tau_y}\mathbf{P}^1\times\mathbf{P}^1\xrightarrow{\text{Segre}}\mathbf{P}^3_{[w:x:y:z]}\xrightarrow{\text{projection}}\mathbf{P}^2_{[x:y:z]}.$$

Choose $x_1,x_2,y_1,y_2\in\mathcal{O}_{\mathbf{P}^{g-1}}(1)$ such that $\tau_x(q)=[x_1(q):x_2(q)]$ and $\tau_y(q)=[y_1(q):y_2(q)]$ in coordinates. Then the equation for φ' is

$$\varphi'\colon q\mapsto[(x_1y_2)(q):(x_2y_1)(q):(x_2y_2)(q)].$$

(When $x_2(q),y_2(q)\neq 0$, this is just $\varphi'\colon q\mapsto[(x_1/x_2)(q):(y_1/y_2)(q):1]$.)

Note that the rational map φ' is not defined on some closed subvarieties of $X_0^+(N)$. Indeed, on the subvariety with additional equations $x_1=x_2=0$, the map τ_x is not defined; with $y_1=y_2=0$, the map τ_y is not defined; with $x_2=y_2=0$, (the map τ_x or τ_y or) the projection following the Segre embedding is not defined. We denote the union of the three subvarieties by

([5]) Note that such τ_x,τ_y always exist: any curve is birational to a curve in $\mathbf{P}^2$ (see [Har77, Corollary IV.3.11]), which is birational to $\mathbf{P}^1\times\mathbf{P}^1$.

$X_0^+(N)_{\varphi' \text{ undef}}$, outside which φ' is well-defined. The set of rational points $X_0^+(N)_{\varphi' \text{ undef}}(\mathbb{Q})$ can be computed directly using Magma [BCP97], and in our final step, we check that it does not contain any additional points other than the known rational points. For now, we focus on the generic part of $X_0^+(N)$ where φ' is defined.

The image $\mathcal{C}'_N := \varphi'(X_0^+(N))$ will be a curve given by an equation of the form

$$Q_0(x,z)y^d + Q_1(x,z)y^{d-1} + \cdots + Q_d(x,z) = 0$$

for some $d \in \mathbf{N}$, where $Q_0(x,1)$ is monic (if not, simply divide the equation throughout by the leading coefficient of $Q_0(x,1)$ to get this form). Multiplying the above equation throughout by Q_0^{d-1}, we get

$$(Q_0(x,z)y)^d + Q_1(x,z)(Q_0(x,z)y)^{d-1} + \cdots + Q_0(x,z)^{d-1}Q_d(x,z) = 0.$$

Let $\deg Q_0$ denote the total degree of Q_0 and let $\mathcal{C}_N$ be the image of $\mathcal{C}'_N$ under the map

$$\psi\colon [x:y:z] \mapsto [xz^{\deg Q_0} : Q_0(x,z)y : z^{\deg Q_0 + 1}].$$

Then the affine patch of $\mathcal{C}_N$ given by $z = 1$ will have an equation $Q(x,y) = 0$ where $Q(x,y)$ is a polynomial over $\mathbb{Q}$ monic in y such that there exists a prime p suitable for the quadratic Chabauty algorithm. Let $\varphi\colon X_0^+(N) \to \mathcal{C}_N$ denote the composite $\varphi = \psi \circ \varphi'$.

Next, we select a suitable prime p and run the `QCModAffine` function on the affine patch of $\mathcal{C}_N$ given by $z = 1$. However, `QCModAffine` only computes rational points (and compares with known rational points) *outside* the *bad* residue disks, which we now define.

Definition 3.1 ([BT20, Definitions 2.8 and 2.10]). Let X^{an} denote the rigid analytic space over $\mathbb{Q}_p$ associated to a projective curve $X/\mathbb{Q}_p$ given by an (inhomogeneous) equation $Q(x,y) = 0$. Let $\Delta(x)$ be the discriminant of Q as a function of y, and let $r(x) = \Delta/\gcd(\Delta, d\Delta/dx)$. We say that a residue disk (as well as any point inside it) is *infinite* if it contains a point whose x-coordinate is ∞, is *bad* if it contains a point whose x-coordinate is ∞ or a zero of $r(x)$, and is *good* if it is not bad.

Therefore, we may need to repeat this construction multiple times (with the same p), i.e., we need a collection $\{(\varphi_{N,i}, \mathcal{C}_{N,i})\}_{i=1}^k$ such that for each $P \in X_0^+(N)(\mathbf{F}_p)$, there exists some i such that $\varphi_{N,i}(P)$ does not map to a bad $\mathbf{F}_p$-point. This would imply that for each residue disk D of $X_0^+(N)(\mathbb{Q}_p)$, there exists some i such that $\varphi_{N,i}(D) \subseteq \mathcal{C}_{N,i}(\mathbb{Q}_p)$ is contained in a good residue disk. If `QCModAffine` reports that the only rational points in the good disks of $\mathcal{C}_{N,i}(\mathbb{Q}_p)$ are the images of the known rational points, then we know that $X_0^+(N)(\mathbb{Q})$ is contained in the finite set

$$\bigcup_{i=1}^{k} \varphi_{N,i}^{-1}\{\varphi_{N,i}(X_0^+(N)(\mathbb{Q})_{\text{known}})\} \cup X_0^+(N)_{\varphi'\,\text{undef}}(\mathbb{Q}),$$

and then we check that this equals $X_0^+(N)(\mathbb{Q})_{\text{known}}$ using direct Magma computations.

In practice, we seek to choose $x_1, x_2, y_1, y_2 \in \mathcal{O}_{\mathbf{P}^{g-1}}(1)$ so that $d_x := \deg \tau_x$ and $d_y := \deg \tau_y$ are small for faster computations, since the degree of the defining equation $Q(x, y)$ of $\mathcal{C}_N$ is at most $d_x d_y$ and the maximum power of y that appears is d_x. We would also like to keep k small, and ideally to have $k = 1$, in which case we need to find a prime p for which $Q(x, y) = 0$ has no bad disks. In order to do so, we would like to start with $Q(x, y)$ for which $r(x)$ has no linear factor over $\mathbb{Q}$. If the construction does not satisfy this property, we adjust the construction of φ' by applying some $\sigma \in \mathrm{Aut}(\mathbf{P}^1 \times_{\mathbb{Q}} \mathbf{P}^1)$ before the Segre embedding, and/or by post-composing φ' by some $\rho \in \mathrm{Aut}(\mathbf{P}^2)$ (the latter only invoked in the $N = 211$ case). With suitable adjustments in the construction of φ', we were able to find plane models $Q(x, y)$ for which $r(x)$ has no linear factor over $\mathbb{Q}$ for all the cases we aim to solve in this paper. Even so, for four of the genus 6 cases, $N = 197, 211, 223, 359$, we did not find a model for which there is a prime p without any bad disks. In each of these four cases, we were able to find two patches and a prime p such that for each $P \in X_0^+(N)(\mathbf{F}_p)$, $\varphi_{N,i}(P)$ does not map to a bad $\mathbf{F}_p$-point on at least one of the patches. For all the other prime levels where $X_0^+(N)$ has genus $4, 5, 6$, we were able to find a single patch and a prime for which there is no bad disk. That is, we were able to solve the problem with $k \le 2$.

In this section, we started with the canonical model of a modular curve in $\mathbf{P}^{g-1}$ (for $g = 4, 5, 6$), but afterwards we did not use the fact that the curve in question has a modular interpretation. This strategy could possibly be applied to other curves of arbitrary genus g. Currently Magma supports computing plane models and gonal maps for curves up to genus 6.

4. Modular curves $X_0^+(N)$ of genus at most 6. In order to find all $X_0^+(N)$ of genus at most 6, we bound the genus of $X_0^+(N)$ below by a strictly increasing function which only depends on N.

Let $g_0(N)$ be the genus of $X_0(N)$ and let $g_0^+(N)$ be the genus of $X_0^+(N)$.

THEOREM 4.1. *For integers $N \ge 1$, we have $g_0(N) \ge (N - 5\sqrt{N} - 8)/12$.*

Proof. See [CWZ00, Section 3]. ∎

Let $\nu(N)$ denote the number of fixed points of w_N. Let $h(D)$ denote the class number of the quadratic order with discriminant D.

THEOREM 4.2. *For $N \geq 5$,*

$$\nu(N) = \begin{cases} h(-4N) + h(-N) & \text{if } N \equiv 3 \pmod 4, \\ h(-4N) & \text{otherwise.} \end{cases}$$

Proof. Apply the formula in [FH99, Remark 2]. ∎

To bound $g_0^+(N)$, we will need an upper bound on $h(D)$. While there are more general bounds (cf. [Lou11, Theorem 3]) on the class number h_K for a number field K, for the imaginary quadratic fields we use the following bound. Such a bound should be well-known, but we include a proof for completeness.

LEMMA 4.3. *For negative D,*

$$h(D) \leq \frac{\sqrt{-D}}{\pi}(\ln(4|D|) + 2).$$

Proof. For negative D, the Dirichlet class number formula states that

$$h(D) = \frac{w\sqrt{-D}}{2\pi} L(1, \chi_D) \tag{4.1}$$

where $\chi_D(m) = \left(\frac{D}{m}\right)$ is the Kronecker symbol and

$$w = \begin{cases} 4 & \text{if } D = -4, \\ 6 & \text{if } D = -3, \\ 2 & \text{otherwise.} \end{cases}$$

Hence, it suffices to obtain an upper bound on $L(1, \chi_D)$. Let $A(x) := \sum_{1 \leq n \leq x} \chi_D(n)$. Since the modulus of χ_D divides $4D$, the sum of any $4|D|$ consecutive values of $\chi_D(n)$ is zero, so

$$|A(x)| \leq 4|D|.$$

Using the definition of $L(1, \chi_D)$ and the Abel summation formula, we get

$$\begin{aligned} L(1, \chi_D) &= \sum_{n=1}^{\infty} \frac{\chi_D(n)}{n} = \sum_{n=1}^{4|D|} \frac{\chi_D(n)}{n} + \sum_{n > 4|D|} \frac{\chi_D(n)}{n} \\ &= \sum_{n=1}^{4|D|} \frac{\chi_D(n)}{n} - \frac{A(4D)}{4D} - \int_{4|D|}^{\infty} \frac{A(t)}{t^2}\, dt. \end{aligned}$$

(Note that the series for $L(1, \chi_D)$ converges because χ_D is non-trivial.) Since $A(4D) = 0$, the middle term vanishes. Using the upper bounds $|\chi_D(n)| \leq 1$ and $|A(x)| \leq 4|D|$ yields

$$L(1, \chi_D) \leq \sum_{n=1}^{4|D|} \frac{1}{n} + 4|D| \int_{4|D|}^{\infty} \frac{1}{t^2}\, dt = \sum_{n=1}^{4|D|} \frac{1}{n} + 1 \leq \ln(4|D|) + 2$$

from the Abel summation formula again. Substituting into the Dirichlet class number formula (4.1) shows that for $D \notin \{-3, -4\}$,

$$h(D) \leq \frac{\sqrt{-D}}{\pi}(\ln(4|D|) + 2).$$

We then manually check that this holds for $D \in \{-3, -4\}$ as well. ∎

Proposition 4.4. *We have*

$$g_0^+(N) \geq \frac{N - 5\sqrt{N} + 4}{24} - \frac{\sqrt{N}}{\pi}(\ln(16N) + 2).$$

Proof. From the Riemann–Hurwitz formula applied to the degree-2 morphism $X_0(N) \to X_0^+(N)$, we have

$$2g_0(N) - 2 = 2(2g_0^+(N) - 2) + \nu(N), \tag{4.2}$$

and we are done by combining Theorems 4.1 and 4.2, and Lemma 4.3. ∎

Proposition 4.5. *The complete list of levels N for which $g_0^+(N) \leq 6$ is given in Table* 1 *for prime N and in Table* 2 *for composite N.*

Table 1. The prime levels N such that $X_0^+(N)$ has genus ≤ 6

$g_0^+(N)$	N
0	$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71$
1	$37, 43, 53, 61, 79, 83, 89, 101, 131$
2	$67, 73, 103, 107, 167, 191$
3	$97, 109, 113, 127, 139, 149, 151, 179, 239$
4	$137, 173, 199, 251, 311$
5	$157, 181, 227, 263$
6	$163, 197, 211, 223, 269, 271, 359$

Table 2. The composite levels N such that $X_0^+(N)$ has genus ≤ 6

$g_0^+(N)$	N
0	$4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 24, 25, 26, 27, 32, 35, 36, 39, 49, 50$
1	$22, 28, 30, 33, 34, 38, 40, 44, 45, 48, 51, 54, 55, 56, 63, 64, 65, 75, 81, 95, 119$
2	$42, 46, 52, 57, 62, 68, 69, 72, 74, 77, 80, 87, 91, 98, 111, 121, 125, 143$
3	$58, 60, 66, 76, 85, 86, 96, 99, 100, 104, 128, 169$
4	$70, 82, 84, 88, 90, 92, 93, 94, 108, 115, 116, 117, 129, 135, 147, 155, 159, 161, 215$
5	$78, 105, 106, 110, 112, 122, 123, 133, 134, 144, 145, 146, 171, 175, 185, 209$
6	$118, 124, 136, 141, 152, 153, 164, 183, 203, 221, 299$

Proof. The lower bound in Proposition 4.4 exceeds 6 for $N > 13300$. For $N \leq 13300$, we compute $g_0^+(N)$ exactly from the computation of $g_0(N)$ in [DS05, § 3.1], Theorem 4.2 (the class numbers of imaginary quadratic fields can be computed efficiently using Magma, e.g. via the identification with the number of equivalence classes of reduced integral binary quadratic forms) and (4.2), and find the N for which $g_0^+(N) \leq 6$. ∎

5. Results and data. In the following tables, we display our choice of models for $X_0^+(N)$ for which we applied `QCMod` successfully. When the rational point is a CM point, we use D to represent the discriminant of the order by which the corresponding elliptic curve has CM. A *Heegner form* (with respect to N and $D < 0$ and subject to the condition that D is a square modulo $4N$) is an integral binary quadratic form $f(x,y) = Ax^2 + Bxy + Cy^2$ of discriminant D, such that the root τ of $f(x,1)$ in the upper-half plane maps to a CM point on $X_0(N)$. There are exactly $h(D)$ Heegner forms of discriminant D. The Heegner forms account for all CM points on $X_0(N)$. (See [Gro84] for more information.)

To compute the coordinates of a CM point of discriminant $D < 0$, we first compute the Heegner forms of elliptic curves with CM by an order of discriminant D. To do so, we use Magma's `HeegnerForms(N,D)` and compute their unique zeros τ with positive imaginary part. Then, we substitute $q = \exp(2\pi i\tau)$ into the q-expansions of the cusp forms giving the canonical embedding, to obtain approximations of the coordinates of the CM point in $\mathbf{P}^{g-1}$. For certain values of D and N, the series converges very slowly. But if there is only one such D for a given N, and all $\mathbb{Q}$-points on $X_0^+(N)$ are known, since we know the modular interpretation for all but one element of $X_0^+(N)(\mathbb{Q})$, we can also determine the interpretation of the remaining one. If all cusp forms vanish at the CM point corresponding to D (this happens for $D = -3, -4$ in the case of $X_0^+(157)$ and $X_0^+(181)$, and $D = -3, -163$ in the case of $X_0^+(163)$), we compute the derivatives of q-expansions of the basis of the cusp forms giving the canonical embedding. In this way, we get the coordinates of the CM points with $D = -4$ and -163, and the only remaining rational point on $X_0^+(157)$ (respectively $X_0^+(181)$) must be the CM point with $D = -3$.

In [BG21, Theorem 2] or [BH03] the triviality of $\mathrm{Aut}(X_0^+(p))$ for our p is proved in general. It implies that the exceptional points on the two curves $X_0^+(137)$ and $X_0^+(311)$ cannot be constructed as an image of a cusp or of a CM point under an automorphism, in contrast to the case for $X_0^+(N)$ hyperelliptic.

We denote by d_∞ the degree of the residue field at infinity. In Table 3 we list p, d_∞, d_x, d_y and the runtime for all affine patches used. The computations were performed on (one core of) a six core server with 64 GB of RAM and processor Intel® Xeon® W-2133 CPU @ 3.60 GHz.

Table 3

N	p	d_∞	d_x	d_y	Runtime
137	5	6	3	3	36 s
173	5	2	4	4	118 s
199	7	4	4	4	143 s
251	11	4	4	4	279 s
311	5	2	4	4	113 s
157	5	4	4	5	372 s
181	7	6	3	5	230 s
227	23	6	3	5	200 s
263	23	4	6	6	1 368 s
163	31	4	4	4	959 s
269	29	4	4	6	2 483 s
271	13	12	5	9	74 142 s
197_1	23	2	6	6	1 450 s
197_2	23	8	5	6	1 129 s
211_1	31	4	6	6	3 523 s
211_2	31	2	6	6	2 092 s
223_1	19	12	6	5	1 565 s
223_2	19	2	6	5	1 137 s
359_1	7	4	6	6	594 s
359_2	7	4	6	5	942 s

For $N = 197, 211, 223, 359$ the two entries in Table 3 correspond to the two affine patches whose equations are presented in Section 5.3.

The models we have used are now listed and organized by their genus. We also write down the classification of all rational points.

The Magma code which can be used to reproduce our computations can be found at GitHub [AAB+21].

5.1. Genus 4

5.1.1. *Data for* $X_0^+(137)$. The equations for the canonical embedding in $\mathbf{P}^{g-1}$ are

$$\begin{aligned}
&XY + WY + 2Y^2 + 2WZ + XZ + 6YZ + 3Z^2 = 0,\\
&X^3 + WX^2 + 6X^2Z - 2XY^2 - 5XYZ + XZW + 13XZ^2 + 2Y^3\\
&\quad + 3WY^2 + W^2Y + 3WYZ - 6YZ^2 + ZW^2 - 4Z^2W + 14Z^3 = 0,
\end{aligned}$$

with variables corresponding to the following cusp forms:

$$\begin{aligned}
W = &-q + q^2 - q^3 + q^4 + 3q^5 + 2q^6 + 4q^7 - 3q^8 + 2q^9 - 5q^{10} - q^{11}\\
&- 2q^{12} + 4q^{13} - 5q^{14} + 3q^{15} + 2q^{16} - 3q^{17} - 2q^{18} - q^{19} + O(q^{20}),
\end{aligned}$$

$$
\begin{aligned}
X &= -q^2 + 2q^3 + 2q^4 - 3q^5 - 2q^6 - 3q^7 + q^8 + q^9 + 4q^{10} + 10q^{11} \\
&\quad - 2q^{12} - q^{13} + 6q^{14} - 4q^{15} - 7q^{16} - 5q^{17} + q^{18} - 7q^{19} + O(q^{20}), \\
Y &= 2q^3 - q^4 - 3q^5 - 3q^6 - q^7 + 3q^8 - 2q^9 + 6q^{10} + 3q^{11} + 3q^{12} - q^{13} \\
&\quad + q^{14} - 3q^{16} + 2q^{17} + 3q^{18} + q^{19} + O(q^{20}), \\
Z &= -q^3 + 2q^5 + 2q^6 + q^7 - q^8 - 3q^{10} - 4q^{11} - q^{12} - q^{14} + 3q^{16} + q^{17} \\
&\quad - 2q^{18} + 2q^{19} + O(q^{20}).
\end{aligned}
$$

The nine known rational points are

$$
\begin{array}{ll}
\text{Cusp}, [1:0:0:0], & D=-16, [2:0:-1:0], \\
D=-4, [2:-4:-3:2], & D=-19, [1:-2:-1:1], \\
D=-7, [2:-1:-2:1], & D=-28, [0:1:2:-1] \\
D=-8, [1:-1:0:0], & \text{Exceptional}, [19:2:-16:4]. \\
D=-11, [1:1:-1:0], &
\end{array}
$$

For quadratic Chabauty, we use $p = 5$ and a single model $(\varphi_{137,1}, \mathcal{C}_{137,1})$. Information for $\varphi_{137,1}$:

$$
\begin{array}{lll}
d_x = 3, & x_1 = Z, & x_2 = Y, \\
d_y = 3, & y_1 = (2\cdot 3\cdot 7)Z, & y_2 = W + X + 2Y + Z.
\end{array}
$$

The equation of $\mathcal{C}_{137,1}$ in the affine patch where $z = 1$ is

$$
\begin{aligned}
&y^3 + (50x^3 + 32x^2 - 4x - 3)y^2 \\
&\quad + (966x^6 + 1377x^5 + 459x^4 - 115x^3 - 66x^2 + x + 2)y \\
&\quad + (7056x^9 + 16128x^8 + 12744x^7 + 2856x^6 - 1239x^5 - 678x^4 \\
&\qquad\qquad - 35x^3 + 28x^2 + 4x) = 0.
\end{aligned}
$$

According to Galbraith [Gal99], the j-invariant of the $\mathbb{Q}$-curve corresponding to the exceptional point is

$$
\begin{aligned}
j = (&-42355484910236534928552761208039609771198 9843 \\
&\pm 9281040308790916967443095886224534005155665\sqrt{-31159})/2^{138}.
\end{aligned}
$$

5.1.2. *Data for $X_0^+(173)$.* The equations for the canonical embedding in $\mathbf{P}^{g-1}$ are

$$
\begin{aligned}
&X^2 + WY + XY - WZ + 4XZ - 3YZ = 0, \\
&XY^2 + Y^3 + W^2Z + WXZ + 2X^2Z + 5WYZ + 6XYZ \\
&\qquad + 9Y^2Z + 3WZ^2 + 11XZ^2 + 14YZ^2 + 12Z^3 = 0,
\end{aligned}
$$

with variables corresponding to the following cusp forms:

$$\begin{aligned} W &= q - 2q^4 - 2q^5 - 3q^7 - 4q^9 + 2q^{11} + q^{12} + 3q^{13} - q^{14} + q^{15} + 3q^{16} \\ &\quad - 3q^{17} - q^{18} - 4q^{19} + O(q^{20}), \\ X &= -q^2 + q^3 + 3q^7 + 4q^8 - q^9 + 2q^{10} - 6q^{11} - q^{12} - 10q^{13} + 2q^{14} \\ &\quad - q^{15} - q^{16} + 7q^{17} + 3q^{18} + 2q^{19} + O(q^{20}), \\ Y &= -q^3 + q^4 + q^5 - q^6 + q^7 + 4q^9 - 4q^{11} - q^{12} - 5q^{13} - q^{15} - 3q^{16} \\ &\quad + 5q^{17} + 4q^{18} + 3q^{19} + O(q^{20}), \\ Z &= q^6 - q^7 - q^8 - q^9 - q^{10} + 3q^{11} + 4q^{13} + q^{16} - 4q^{17} - 3q^{18} \\ &\quad - 2q^{19} + O(q^{20}). \end{aligned}$$

The six known rational points are

$$\begin{array}{ll} \text{Cusp}, [1:0:0:0], & D = -43, [0:1:-1:0], \\ D = -4, [0:-4:0:1], & D = -67, [3:-3:-2:1], \\ D = -16, [2:-2:-2:1], & D = -163, [12:-9:-5:2]. \end{array}$$

For quadratic Chabauty, we use $p = 5$ and a single model $(\varphi_{173,1}, \mathcal{C}_{173,1})$. Information for $\varphi_{173,1}$:

$$\begin{array}{lll} d_x = 4, & x_1 = W + X + 2Y + 4Z, & x_2 = W + Y, \\ d_y = 4, & y_1 = (3 \cdot 7/2) \cdot (X - 3Y - 3Z), & y_2 = 2W + X + 9Y + 15Z. \end{array}$$

The equation of $\mathcal{C}_{173,1}$ in the affine patch where $z = 1$ is

$$\begin{aligned} & y^4 + \frac{1}{2^2 \cdot 3}(198x^4 - 51x^3 + 8x^2 - 495x + 54)y^3 \\ & + \frac{1}{2^5 \cdot 3^2} \begin{pmatrix} 33264x^8 - 15804x^7 + 1164x^6 - 166127x^5 + 81837x^4 \\ - 18480x^3 + 180184x^2 - 44117x + 1743 \end{pmatrix} y^2 \\ & + \frac{1}{2^8 \cdot 3^3} \begin{pmatrix} 2667168x^{12} - 1333584x^{11} - 1775088x^{10} \\ - 17635032x^9 + 14542866x^8 - 2087801x^7 \\ + 40446120x^6 - 27642183x^5 + 6246796x^4 \\ - 27504295x^3 + 11701160x^2 - 957873x + 22050 \end{pmatrix} y \\ & + \frac{1}{2^{12} \cdot 3^4} \begin{pmatrix} 192036096x^{16} - 132596352x^{15} - 350759808x^{14} \\ - 1097128800x^{13} + 1384863264x^{12} + 522223848x^{11} \\ + 3767950920x^{10} - 4672257034x^9 + 908642397x^8 \\ - 5781010796x^7 + 5563290280x^6 - 1437899280x^5 \\ + 2910049418x^4 - 2024404284x^3 + 258937560x^2 \\ - 12131910x + 194481 \end{pmatrix} = 0. \end{aligned}$$

5.1.3. *Data for $X_0^+(199)$.* The equations for the canonical embedding in $\mathbf{P}^{g-1}$ are

$$\begin{aligned}
&2WY - X^2 + 3XY - 6XZ - 5WZ + 3YZ - 6Z^2 = 0,\\
&-WX^2 + 3W^2Y + 3WXY + 3WY^2 + XY^2 + Y^3 - 7W^2Z - 5WXZ\\
&\qquad - 8WYZ - 2XYZ - 3Y^2Z + 6WZ^2 + 4YZ^2 - 3Z^3 = 0,
\end{aligned}$$

with variables corresponding to the following cusp forms:

$$\begin{aligned}
W &= q + q^3 - 2q^4 - 2q^5 - q^6 - 6q^7 - 3q^9 + q^{10} + 2q^{11} - q^{12} + 3q^{13} + 2q^{14}\\
&\quad + 5q^{16} - q^{17} + q^{18} - 3q^{19} + O(q^{20}),\\
X &= q^2 + q^3 - 3q^5 - 3q^6 - 2q^7 - 4q^8 - q^9 + 2q^{10} + 3q^{11} + q^{12} + 5q^{13} + 2q^{14}\\
&\quad + 3q^{15} + 4q^{16} + 4q^{17} + q^{18} - 4q^{19} + O(q^{20}),\\
Y &= -2q^3 + q^4 + q^5 + q^6 + 5q^7 + 2q^9 - 3q^{10} - 4q^{11} + q^{12} - 3q^{13} - 2q^{14}\\
&\quad + q^{15} - 6q^{16} - 3q^{17} - q^{18} + 5q^{19} + O(q^{20}),\\
Z &= -q^3 + q^5 + q^6 + 2q^7 + q^8 + q^9 - 2q^{10} - 2q^{11} - 2q^{13} - q^{14} - 3q^{16} - 2q^{17}\\
&\quad - q^{18} + 3q^{19} + O(q^{20}).
\end{aligned}$$

The eight known rational points are

$$\begin{array}{ll}
\text{Cusp, } [1:0:0:0], & D = -19,\ [1:-1:-1:0],\\
D = -3,\ [3:-3:-3:-1], & D = -27,\ [3:0:-3:-1],\\
D = -11,\ [1:1:-2:-1], & D = -67,\ [4:5:-3:-2],\\
D = -12,\ [1:3:-1:-1], & D = -163,\ [5:-2:-3:-1].
\end{array}$$

For quadratic Chabauty, we use $p = 7$ and a single model $(\varphi_{199,1}, \mathcal{C}_{199,1})$. Information for $\varphi_{199,1}$:

$$\begin{array}{lll}
d_x = 4, & x_1 = W + Y, & x_2 = W - X + 2Y - 3Z,\\
d_y = 4, & y_1 = (5 \cdot 11) \cdot (Y - Z), & y_2 = X - 4Y + 7Z.
\end{array}$$

The equation of $\mathcal{C}_{199,1}$ in the affine patch where $z = 1$ is

$$\begin{aligned}
&y^4 + \left(188x^4 - 531x^3 + 730x^2 - 585x + 195\right) y^3\\
&\quad + \begin{pmatrix} 6930x^8 - 36467x^7 + 102801x^6 - 201149x^5\\ + 280108x^4 - 273201x^3 + 179227x^2 - 70731x + 12474 \end{pmatrix} y^2\\
&\quad + \begin{pmatrix} 87725x^{12} - 678755x^{11} + 2812416x^{10} - 8175060x^9\\ + 17971418x^8 - 30862293x^7 + 42039858x^6 - 45235151x^5\\ + 37608990x^4 - 23234260x^3 + 9991212x^2 - 2649537x + 323433 \end{pmatrix} y
\end{aligned}$$

$$+\begin{pmatrix} 332750x^{16} - 3454550x^{15} + 18928470x^{14} - 72428151x^{13} \\ + 212751798x^{12} - 501847128x^{11} + 975020037x^{10} \\ - 1580007615x^{9} + 2144054322x^{8} - 2430428083x^{7} \\ + 2279776105x^{6} - 1737806496x^{5} + 1046077245x^{4} \\ - 476258451x^{3} + 153252000x^{2} - 30873150x + 2910897 \end{pmatrix} = 0.$$

5.1.4. *Data for* $X_0^+(251)$. The equations for the canonical embedding in $\mathbf{P}^{g-1}$ are

$$\begin{aligned} &WX - WY + XY - 2Y^2 + 2WZ + XZ + 4YZ + Z^2 = 0, \\ &W^2X - W^2Y + X^2Y - WY^2 - 2XY^2 + 2Y^3 + W^2Z - WXZ \\ &\quad - 3X^2Z + 2WYZ + 7XYZ - 3Y^2Z - 21XZ^2 + 10YZ^2 - 28Z^3 = 0, \end{aligned}$$

with variables corresponding to the following cusp forms:

$$\begin{aligned} W &= -q - q^2 + 2q^4 + 3q^5 + q^7 + 3q^8 + 3q^9 + 2q^{10} - q^{11} + q^{13} + q^{14} + q^{15} \\ &\quad - 3q^{16} - 3q^{17} + 2q^{18} + 4q^{19} + O(q^{20}), \\ X &= q^2 - q^3 - q^4 + 2q^5 - q^6 - q^7 - 2q^8 + q^{10} - q^{11} + 2q^{12} - q^{14} + 3q^{16} \\ &\quad - 6q^{17} + 4q^{19} + O(q^{20}), \\ Y &= q^2 - q^4 - 2q^5 + q^6 + q^7 - 2q^8 - q^9 - 2q^{10} + 2q^{11} - q^{12} + q^{13} - 2q^{14} \\ &\quad + 3q^{16} + 4q^{17} - 2q^{18} - 4q^{19} + O(q^{20}), \\ Z &= -q^5 + q^6 + q^7 - q^{10} + q^{11} - q^{12} - q^{14} + 3q^{17} - q^{18} - 3q^{19} + O(q^{20}). \end{aligned}$$

The six known rational points are

$$\begin{array}{ll} \text{Cusp}, [1:0:0:0], & D=-19, [2:1:-2:-1], \\ D=-8, [1:2:-1:-1], & D=-43, [2:0:-1:0], \\ D=-11, [0:1:0:0], & D=-163, [16:-3:-6:1]. \end{array}$$

For quadratic Chabauty, we use $p = 11$ and a single model $(\varphi_{251,1}, \mathcal{C}_{251,1})$. Information for $\varphi_{251,1}$:

$$\begin{array}{lll} d_x = 4, & x_1 = W + 7X + 5Z, & x_2 = W - 6X + 10Y - 23Z, \\ d_y = 4, & y_1 = (5/3)Z, & y_2 = X - Z. \end{array}$$

The equation of $\mathcal{C}_{251,1}$ in the affine patch where $z = 1$ is

$$\begin{aligned} &y^4 + \frac{1}{2^2\cdot 3\cdot 5}(235x^4 - 1400x^3 - 3188x^2 - 1402x - 330)y^3 \\ &\quad + \frac{1}{2^4\cdot 3^2\cdot 5^2}\begin{pmatrix} 16700x^8 - 247290x^7 + 176078x^6 \\ + 2948981x^5 + 4658866x^4 + 3074269x^3 \\ + 1328589x^2 + 339250x + 42800 \end{pmatrix} y^2 \end{aligned}$$

$$+\frac{1}{2^6\cdot 3^3\cdot 5^3}\begin{pmatrix} 510000x^{12}-12344000x^{11}+61126700x^{10} \\ +217315540x^9-425974544x^8-2004263968x^7 \\ -2611092185x^6-1852500895x^5-1038555014x^4 \\ -475857025x^3-154375600x^2-29480000x-2560000 \end{pmatrix} y$$

$$+\frac{1}{2^8\cdot 3^4\cdot 5^4}\begin{pmatrix} 6000000x^{16}-203600000x^{15}+1956240000x^{14} \\ -664896000x^{13}-41347955400x^{12}-66346728360x^{11} \\ +59428683804x^{10}+181499926316x^9+43955025678x^8 \\ -137005175370x^7-105387676043x^6+907612425x^5 \\ +35725442325x^4+20724635625x^3+5829465000x^2 \\ +867000000x+56000000 \end{pmatrix} = 0.$$

5.1.5. *Data for* $X_0^+(311)$. The equations for the canonical embedding in $\mathbf{P}^{g-1}$ are

$$\begin{aligned} &X^2+WY-2XY+2Y^2+7XZ-8YZ+13Z^2=0,\\ &WX^2-2WXY+X^2Y-WY^2-XY^2-2Y^3+W^2Z+6WXZ\\ &\quad -X^2Z-WYZ+5XYZ+4Y^2Z+7WZ^2-4XZ^2-2Z^3=0, \end{aligned}$$

with variables corresponding to the following cusp forms:

$$\begin{aligned} W &= q-2q^4-q^5+q^6-2q^9-q^{10}-q^{11}-3q^{13}-q^{14}-2q^{15}+3q^{16}-q^{18}\\ &\quad -q^{19}+O(q^{20}),\\ X &= q^2+2q^3-3q^5-q^6-3q^7-4q^8-2q^9-3q^{12}+q^{13}+q^{14}+q^{15}+q^{16}\\ &\quad +7q^{17}-q^{18}+2q^{19}+O(q^{20}),\\ Y &= -q^3+q^4+q^5-q^6+q^9+q^{12}+q^{13}+q^{15}-3q^{16}-3q^{17}+q^{18}-q^{19}\\ &\quad +O(q^{20}),\\ Z &= -q^3+q^5+q^7+q^8+q^9+q^{12}-q^{14}-q^{16}-3q^{17}-q^{19}+O(q^{20}). \end{aligned}$$

The five known rational points are

Cusp, $[1:0:0:0]$,	$D=-43$, $[2:0:-1:0]$,
$D=-11$, $[1:-1:-1:0]$,	Exceptional, $[6:8:-1:-2]$.
$D=-19$, $[1:2:-1:-1]$,	

For quadratic Chabauty, we use $p=5$ and a single model $(\varphi_{311,1},\mathcal{C}_{311,1})$. Information for $\varphi_{311,1}$:

$$\begin{aligned} &d_x=4, && x_1=W+X+Z, && x_2=W+Y-Z,\\ &d_y=4, && y_1=(3\cdot 7)Z, && y_2=W+X+Y. \end{aligned}$$

The equation of $\mathcal{C}_{311,1}$ in the affine patch where $z=1$ is

$$\begin{aligned} & y^4 + (48x^4 - 140x^3 + 74x^2 + 23x - 1)y^3 \\ & \quad + \begin{pmatrix} 903x^8 - 5223x^7 + 10128x^6 - 6291x^5 - 1761x^4 + 2334x^3 + 90x^2 \\ - 360x + 72 \end{pmatrix} y^2 \\ & \quad + \begin{pmatrix} 7938x^{12} - 68229x^{11} + 224046x^{10} - 327762x^9 + 135747x^8 + 159903x^7 \\ - 147780x^6 - 33084x^5 + 58905x^4 + 1044x^3 - 11340x^2 + 144x + 1044 \end{pmatrix} y \\ & \quad + \begin{pmatrix} 27783x^{16} - 318843x^{15} + 1488564x^{14} - 3490182x^{13} + 3760074x^{12} \\ + 40986x^{11} - 3866940x^{10} + 2042064x^9 + 1845207x^8 - 1526499x^7 \\ - 580176x^6 + 553014x^5 + 152064x^4 - 109188x^3 - 30240x^2 \\ + 9288x + 3024 \end{pmatrix} = 0. \end{aligned}$$

According to Galbraith [Gal99], the j-invariant of the $\mathbb{Q}$-curve corresponding to the exceptional point is

$$\begin{aligned} j = & 3124418359443327073099098579305858972915260167782400000 \\ & \pm 156581053899805171539733968949219503507755126784000 0\sqrt{39816211853}. \end{aligned}$$

5.2. Genus 5

5.2.1. *Data for* $X_0^+(157)$. The equations for the canonical embedding in $\mathbf{P}^{g-1}$ are

$$\begin{aligned} & VX - VY + WY - XY + YZ = 0, \\ & V^2 - VW + WX + WY + XY + VZ - WZ + XZ = 0, \\ & 2VX + WX - VY - XY - VZ - WZ + 2XZ - Z^2 = 0, \end{aligned}$$

with variables corresponding to the following cusp forms

$$\begin{aligned} V &= -q^4 + q^5 + q^6 + 2q^8 - q^9 - 2q^{10} - 2q^{11} - q^{13} + 2q^{14} - q^{15} - q^{17} - q^{18} \\ & \quad + 3q^{19} + O(q^{20}), \\ W &= -q + q^2 + q^3 - q^4 + q^5 + q^7 + 2q^8 - q^{10} + 2q^{11} + q^{13} + q^{15} - q^{16} \\ & \quad + 3q^{17} - q^{18} + q^{19} + O(q^{20}), \\ X &= -q^2 + q^3 + q^4 + q^8 - 2q^9 + 2q^{10} + q^{12} - q^{13} + 2q^{14} - 2q^{15} - q^{16} - 4q^{17} \\ & \quad + 2q^{18} + q^{19} + O(q^{20}), \\ Y &= q^5 - q^6 - 2q^7 + q^8 + q^{11} + 2q^{12} + q^{13} + q^{14} - q^{15} - 3q^{16} - 2q^{17} + 3q^{18} \\ & \quad - q^{19} + O(q^{20}), \\ Z &= -q^2 + 2q^4 + q^5 + q^6 - q^7 - q^8 + q^9 + q^{10} + q^{11} - 2q^{12} - 2q^{15} - 2q^{17} \\ & \quad - q^{18} - 2q^{19} + O(q^{20}). \end{aligned}$$

The nine known rational points are

$$\begin{array}{ll}
\text{Cusp}, [0:1:0:0:0], & D=-16, [1:1:0:0:0], \\
D=-3, [3:2:0:-2:1], & D=-19, [0:0:1:0:0], \\
D=-4, [1:1:0:0:-2], & D=-27, [0:1:0:-1:-1], \\
D=-11, [0:0:0:1:0], & D=-67, [1:2:-1:1:-1]. \\
D=-12, [1:0:0:0:-1], &
\end{array}$$

For quadratic Chabauty, we use $p=5$ and a single model $(\varphi_{157,1}, \mathcal{C}_{157,1})$. Information for $\varphi_{157,1}$:

$$\begin{array}{lll}
d_x = 4, & x_1 = 3(V+2Y+4Z), & x_2 = V-4W-3X-2Y-5Z, \\
d_y = 5, & y_1 = V-W+Y, & y_2 = 2V-2W-Y.
\end{array}$$

The equation of $\mathcal{C}_{157,1}$ in the affine patch where $z=1$ is

$$\begin{aligned}
y^4 &+ \frac{1}{2\cdot 3^2}(-52x^5-683x^4-3544x^3-9531x^2-12978x-6885)y^3 \\
&+ \frac{1}{2^2\cdot 3^3}\begin{pmatrix} 234x^{10}+6461x^9+80107x^8+589432x^7+2842712x^6+9365724x^5 \\ +21342438x^4+33232032x^3+33792228x^2+20187468x+5353776 \end{pmatrix} y^2 \\
&+ \frac{1}{2^3\cdot 3^6}\begin{pmatrix} 972x^{15}+8640x^{14}-474801x^{13}-13058977x^{12}-162437862x^{11} \\ -1259359425x^{10}-6767639914x^9-26534799963x^8-77953645944x^7 \\ -173540777940x^6-292464845748x^5-368070954012x^4 \\ -335685400200x^3-209644813980x^2-80154057384x-14129953308 \end{pmatrix} y \\
&+ \frac{1}{2^4\cdot 3^8}\begin{pmatrix} -40824x^{20}-1958580x^{19}-42950790x^{18}-571046195x^{17} \\ -5137341277x^{16}-32912831723x^{15}-152493625145x^{14} \\ -496440499072x^{13}-965538929930x^{12}+126852890994x^{11} \\ +9026198316090x^{10}+38983887443916x^9+105405734310336x^8 \\ +208444113891396x^7+314293743788076x^6+363977212778208x^5 \\ +319893174321048x^4+206961055983144x^3 \\ +93048452882280x^2+25964710722144x+3383816644368 \end{pmatrix} = 0.
\end{aligned}$$

Galbraith has a small typo in his model for $X_0^+(157)$. His third equation should have leading coefficient $2w^2$ instead of w^2. The following linear map sends our model to his model:

$$\begin{aligned}
v &= 3V-W-X+Y+Z, \\
w &= 2V-2X-2Y+Z, \\
x &= -4V+3X-Y-3Z, \\
y &= -V+X+Y-Z, \\
z &= V-X+Z,
\end{aligned}$$

and for reference, the corrected Galbraith equations should be as follows:

$$
\begin{aligned}
&-vx+2wx-5vy-8vz+5wz-2xz+5yz+z^2=0,\\
&w^2+wx+4wy-xy+y^2+vz+4wz-2xz-9yz-12z^2=0,\\
&\mathbf{2}w^2+3wx+x^2-vy+9wy+2xy+6y^2+vz+12wz+5xz+3yz-z^2=0.
\end{aligned}
$$

5.2.2. *Data for* $X_0^+(181)$. The equations for the canonical embedding in $\mathbf{P}^{g-1}$ are

$$
\begin{aligned}
&XY+Y^2-WZ+XZ+3YZ+4Z^2=0,\\
&VW+3WX+3VY+WY+4XY-2Y^2\\
&\quad +VZ+16WZ-2XZ+18YZ-10Z^2=0,\\
&VX+3X^2+VY-6XY-9Y^2+5VZ\\
&\quad +10WZ+16XZ-14YZ+14Z^2=0,\\
&V^2W+VW^2-W^3+3VWX+W^2X-VX^2-2X^3-V^2Y+VWY\\
&\quad -5W^2Y+3VXY-2WXY+4X^2Y+VY^2-3WY^2+10Y^3\\
&\quad -3V^2Z-VWZ+2W^2Z-7VXZ-15WXZ-9X^2Z+2VYZ\\
&\quad -WYZ-3XYZ-8Y^2Z+VZ^2+4WZ^2-XZ^2+4YZ^2-2Z^3=0,\\
&V^2W+VW^2-W^3+V^2X+3VWX+W^2X+3VX^2+X^3+2V^2Y\\
&\quad +3VWY-7W^2Y+8WXY-6VY^2-7WY^2+XY^2\\
&\quad -2Y^3+4V^2Z+14VWZ-8W^2Z+14VXZ+5X^2Z\\
&\quad -5VYZ+4WYZ-XYZ+4Y^2Z+6VZ^2-6WZ^2+3XZ^2=0,
\end{aligned}
$$

with variables corresponding to the following cusp forms:

$$
\begin{aligned}
V&=q+2q^2-3q^3-q^4-q^5-5q^6-3q^7-2q^8+6q^9-3q^{10}-6q^{11}+q^{12}\\
&\quad +5q^{13}-q^{14}+3q^{15}-4q^{16}-q^{17}+8q^{18}+2q^{19}+O(q^{20}),\\
W&=2q^2-5q^4+2q^5-2q^6-4q^7+4q^8-5q^{10}-q^{11}+6q^{12}-4q^{13}+9q^{14}\\
&\quad -5q^{15}+2q^{16}+q^{17}-3q^{18}+4q^{19}+O(q^{20}),\\
X&=-4q^2+q^3+3q^4+3q^5+8q^6+4q^7+q^8-6q^9+4q^{10}+q^{11}-3q^{12}\\
&\quad -9q^{13}-5q^{14}-5q^{15}+5q^{16}+3q^{17}-6q^{18}-10q^{19}+O(q^{20}),\\
Y&=-q^2+2q^4+q^6+q^7-q^8+2q^{10}-2q^{12}+q^{13}-3q^{14}+q^{15}-q^{16}+q^{18}\\
&\quad -2q^{19}+O(q^{20}),\\
Z&=q^2-q^4-q^5-2q^6-q^7+q^9-q^{10}+q^{12}+2q^{13}+2q^{14}+q^{15}-q^{16}-q^{17}\\
&\quad +q^{18}+3q^{19}+O(q^{20}).
\end{aligned}
$$

The nine known rational points are

$$\begin{array}{ll}
\text{Cusp}, [1:0:0:0:0], & D=-16, [2:-2:-1:1:0], \\
D=-3, [13:9:-11:-3:2], & D=-27, [2:0:-4:0:1], \\
D=-4, [2:2:-7:-1:2], & D=-43, [3:1:-1:0:0], \\
D=-11, [2:-3:-1:1:0], & D=-67, [2:6:-4:-2:1]. \\
D=-12, [1:-3:-1:1:0], &
\end{array}$$

For quadratic Chabauty, we use $p = 7$ and a single model $(\varphi_{181,1}, \mathcal{C}_{181,1})$. Information for $\varphi_{181,1}$:

$$\begin{aligned}
d_x &= 3, \quad x_1 = Z, \quad x_2 = X + Y + 8Z, \\
d_y &= 5, \quad y_1 = (2 \cdot 17 \cdot 127) \cdot (V + W + X + 2Y + 2Z), \\
& \qquad\;\; y_2 = 2V - W + 3X - 6Y + 5Z.
\end{aligned}$$

The equation of $\mathcal{C}_{181,1}$ in the affine patch where $z = 1$ is

$$\begin{aligned}
& y^3 + (-11088x^5 + 16124x^4 - 9178x^3 + 2548x^2 - 344x + 18)y^2 \\
& + \begin{pmatrix} 37937948x^{10} - 109109424x^9 + 139717096x^8 - 104840620x^7 \\ + 51026716x^6 - 16825292x^5 + 3805736x^4 \\ - 583156x^3 + 57968x^2 - 3380x + 88 \end{pmatrix} y \\
& + \begin{pmatrix} -38931018912x^{15} + 165273816264x^{14} - 324830859688x^{13} \\ + 391968536592x^{12} - 324676043144x^{11} + 195506522152x^{10} \\ - 88396329808x^9 + 30555659584x^8 - 8141070384x^7 \\ + 1672000152x^6 - 262592968x^5 + 30981584x^4 \\ - 2659496x^3 + 156920x^2 - 5696x + 96 \end{pmatrix} = 0.
\end{aligned}$$

5.2.3. *Data for $X_0^+(227)$.* The equations for the canonical embedding in $\mathbf{P}^{g-1}$ are

$$\begin{aligned}
& 3VW + 2W^2 - VX + WX + X^2 + 3VY + 11WY + XY + 2Y^2 \\
& \quad - 5VZ - 7WZ - 7XZ - 7YZ + 3Z^2 = 0, \\
& VW + 3W^2 + VX + 2WX - X^2 - VY \\
& \quad + 4XY - 3Y^2 - VZ + 2WZ - 8XZ + 13YZ - 12Z^2 = 0, \\
& 2V^2 + 3VW + 5W^2 + 2VX + WX - 4X^2 + 2VY - 7WY \\
& \quad + 2XY + 2Y^2 - 2VZ + 12WZ - 10XZ - 7YZ + 3Z^2 = 0, \\
& 2W^3 + 2V^2X - VWX - 3WX^2 - 2X^3 - 2V^2Y - 4VWY \\
& \quad - W^2Y - VXY - 6WXY - X^2Y - VY^2 - Y^3 + 3V^2Z \\
& \quad - VWZ - 2W^2Z + 2VXZ - 4WXZ + 3VYZ - 6WYZ \\
& \quad - 3XYZ + 2Y^2Z - 3VZ^2 - WZ^2 + 5XZ^2 - YZ^2 - 2Z^3 = 0,
\end{aligned}$$

$$\begin{aligned}
&V^3+V^2W+5VW^2+2W^3-VWX-W^2X-2VX^2-WX^2+X^3\\
&\quad -3VWY-W^2Y+VXY+WXY-2X^2Y-VY^2-3WY^2\\
&\quad -XY^2-Y^3-V^2Z+3VWZ+8W^2Z-VXZ-4WXZ+5X^2Z\\
&\quad +VYZ+6XYZ+2Y^2Z+3WZ^2-3XZ^2+4YZ^2-3Z^3=0,
\end{aligned}$$

with variables corresponding to the following cusp forms:

$$\begin{aligned}
V&=-q+2q^3+q^7-q^9+2q^{10}-q^{11}+4q^{13}+4q^{14}+4q^{16}+4q^{17}\\
&\quad -5q^{19}+O(q^{20}),\\
W&=q^2-q^5-2q^6-2q^7-2q^8+2q^{11}+2q^{13}-q^{14}+2q^{15}+q^{17}+q^{18}\\
&\quad +2q^{19}+O(q^{20}),\\
X&=-q-q^3+2q^4+3q^5+q^6+2q^7-q^8-q^{10}+3q^{11}+2q^{12}+3q^{13}-q^{14}\\
&\quad +4q^{15}-2q^{16}-3q^{17}+q^{18}+O(q^{20}),\\
Y&=-q^2+2q^3-q^5-3q^7+3q^8+q^9+2q^{10}+q^{11}-3q^{12}+q^{14}-5q^{15}+q^{16}\\
&\quad -q^{17}+q^{18}+7q^{19}+O(q^{20}),\\
Z&=q^3-q^4-q^5-q^7+2q^8+q^9+q^{10}-q^{11}-2q^{12}-q^{14}-3q^{15}+q^{16}\\
&\quad -q^{18}+4q^{19}+O(q^{20}).
\end{aligned}$$

The four known rational points are

$$\begin{aligned}
&\text{Cusp}, [1:0:1:0:0], && D=-67, [1:2:-1:-1:-1],\\
&D=-8, [1:0:0:-1:-1], && D=-163, [1:-4:-7:5:2].
\end{aligned}$$

For quadratic Chabauty, we use $p=23$ and a single model $(\varphi_{227,1},\mathcal{C}_{227,1})$. Information for $\varphi_{227,1}$:

$$\begin{aligned}
&d_x=3, && x_1=7(W+X-Y+5Z), && x_2=2V+6W+5X-Y+17Z,\\
&d_y=5, && y_1=(7\cdot 17)Z, && y_2=2V+6W+5X-Y+17Z.
\end{aligned}$$

The equation of $\mathcal{C}_{227,1}$ in the affine patch where $z=1$ is

$$\begin{aligned}
&y^3+(-16x^3+49x^2-41x+5)y^2\\
&\quad +(102x^6-668x^5+1787x^4-2536x^3+2094x^2-1027x+252)y\\
&\quad +\begin{pmatrix}-289x^9+3111x^8-15134x^7+44025x^6-85025x^5\\ +113565x^4-104896x^3+64299x^2-23544x+3888\end{pmatrix}=0.
\end{aligned}$$

5.2.4. *Data for $X_0(263)^+$.* The equations for the canonical embedding in $\mathbf{P}^{g-1}$ are

$$\begin{aligned}
&VW+2X^2+WY+2XY-2XZ-YZ+Z^2=0,\\
&VX+2X^2+VY+WY+XY-VZ-2XZ-YZ+Z^2=0,\\
&WX-3X^2-VY-2WY-4XY-VZ+4XZ+3YZ-2Z^2=0,
\end{aligned}$$

with variables corresponding to the following cusp forms:

$$\begin{aligned}
V &= q^6 - q^7 - q^8 + q^{11} - q^{12} + q^{13} + q^{16} + q^{17} - 2q^{18} + 2q^{19} + O(q^{20}),\\
W &= -q + q^3 + q^5 + q^6 + q^7 + q^{10} + q^{12} + 3q^{13} + q^{14} - q^{15} + 2q^{16} + 2q^{17}\\
&\quad - q^{18} + q^{19} + O(q^{20}),\\
X &= q^4 - q^5 - q^6 + q^9 + q^{11} - q^{12} - q^{13} + q^{15} - 3q^{16} + 2q^{17} + q^{18} + O(q^{20}),\\
Y &= q^3 - q^4 - q^7 - 2q^9 + q^{10} - q^{11} + q^{12} + q^{13} - q^{15} + 2q^{16} + 2q^{17}\\
&\quad + q^{19} + O(q^{20}),\\
Z &= q^2 - q^5 - 2q^6 - q^7 - q^8 - q^{10} + q^{13} - q^{14} + 2q^{15} - q^{16} + q^{17} + 2q^{18}\\
&\quad + 2q^{19} + O(q^{20}).
\end{aligned}$$

The six known rational points are

$$\begin{array}{ll}
\text{Cusp}, [0:1:0:0:0], & D=-28, [1:0:0:2:2],\\
D=-7, [1:0:0:0:0], & D=-67, [1:0:1:-1:1],\\
D=-19, [0:0:0:1:0], & D=-163, [2:1:3:-4:4].
\end{array}$$

For quadratic Chabauty, we use $p = 23$ and a single model $(\varphi_{263,1}, \mathcal{C}_{263,1})$. Information for $\varphi_{263,1}$:

$$\begin{aligned}
d_x &= 6, \quad x_1 = V - 2X + Z, \quad x_2 = 2V - W - X,\\
d_y &= 6, \quad y_1 = W + X - Z, \quad y_2 = 2V - W - X.
\end{aligned}$$

The equation of $\mathcal{C}_{263,1}$ in the affine patch where $z = 1$ is

$$\begin{aligned}
&y^6 + \frac{1}{2}(-4x+9)\,y^5 + \frac{1}{2}\left(7x^2 - 24x + 24\right) y^4\\
&\quad + \frac{1}{2}\left(-6x^3 + 19x^2 - 28x + 28\right) y^3 + \frac{1}{2}\left(7x^4 - 31x^3 + 41x^2 - 19x + 13\right) y^2\\
&\quad + \frac{1}{2}\left(-2x^5 + 9x^4 - 21x^3 + 25x^2 - 9x + 2\right) y\\
&\quad + \frac{1}{2}\left(2x^6 - 9x^5 + 14x^4 - 9x^3 + 4x^2 - x\right) = 0.
\end{aligned}$$

5.3. Genus 6

5.3.1. *Data for $X_0^+(163)$.* The equations for the canonical embedding in $\mathbf{P}^{g-1}$ are

$$\begin{aligned}
&VX - UY - XY = 0,\\
&UY - WY - VZ = 0,\\
&UX - WX - UZ - XZ = 0,\\
&UX + VX + VY + VZ - YZ - Z^2 = 0,\\
&UV + V^2 - VY + WY + WZ + XZ = 0,\\
&U^2 + UV - VW + WX + VY + XY = 0,
\end{aligned}$$

with variables corresponding to the following cusp forms:

$$\begin{aligned}
U &= -q^2 + q^3 + q^4 - q^9 + 3q^{10} - 2q^{11} + 2q^{12} + 2q^{14} - 4q^{15} - 2q^{18} \\
&\quad + q^{19} + O(q^{20}), \\
V &= -q^3 + q^4 + q^5 + 2q^9 - 2q^{10} + q^{11} - 2q^{12} - 3q^{13} - q^{14} + 3q^{15} - 3q^{16} \\
&\quad - q^{17} + 3q^{18} + 4q^{19} + O(q^{20}), \\
W &= -q + q^3 + q^4 + 2q^5 + q^7 + q^8 + q^9 + q^{10} + q^{12} + q^{14} - 2q^{15} - 3q^{16} \\
&\quad + 3q^{17} - q^{18} + 2q^{19} + O(q^{20}), \\
X &= -q^5 + q^6 + 2q^7 - q^8 - 2q^{11} - 2q^{12} + 2q^{13} - q^{14} + 3q^{16} + q^{17} - 2q^{18} \\
&\quad - 4q^{19} + O(q^{20}), \\
Y &= -q^6 + q^7 + 2q^8 - q^{10} - 3q^{11} + q^{12} - q^{13} + q^{14} + q^{15} - 5q^{16} + q^{17} \\
&\quad + 4q^{18} + 4q^{19} + O(q^{20}), \\
Z &= q^4 - q^5 - q^6 - 2q^8 + 2q^9 + q^{10} + q^{11} + q^{14} + 2q^{16} - q^{18} - q^{19} + O(q^{20}).
\end{aligned}$$

The eleven known rational points are

$$\begin{array}{ll}
\text{Cusp},\ [0:0:1:0:0:0], & D=-19,\ [1:-1:0:0:0:0], \\
D=-3,\ [0:1:-2:3:1:2], & D=-27,\ [0:1:1:0:1:-1], \\
D=-7,\ [0:0:0:0:1:-1], & D=-67,\ [1:0:-1:1:0:1], \\
D=-8,\ [0:0:0:1:0:0], & D=-28,\ [2:-2:0:2:-1:1], \\
D=-11,\ [0:0:0:0:1:0], & D=-163,\ [78:-50:-42:-13:10:-24]. \\
D=-12,\ [0:1:0:-1:1:0], &
\end{array}$$

For quadratic Chabauty, we use $p = 31$ and a single model $(\varphi_{163,1}, \mathcal{C}_{163,1})$. Information for $\varphi_{163,1}$:

$$\begin{aligned}
d_x &= 4, \quad x_1 = Y + Z, \quad x_2 = 2X - Y - Z, \\
d_y &= 4, \quad y_1 = V - Y, \quad y_2 = V + Y.
\end{aligned}$$

The equation of $\mathcal{C}_{163,1}$ in the affine patch where $z = 1$ is

$$\begin{aligned}
y^4 &- \frac{1}{2^3 \cdot 5}(x^4 + 16x^3 + 22x^2 + 8x + 1)y^3 \\
&- \frac{1}{2^6 \cdot 5^2} \begin{pmatrix} 2920x^8 + 9243x^7 + 13419x^6 + 11947x^5 \\ + 7067x^4 + 2873x^3 + 777x^2 + 129x + 9 \end{pmatrix} y^2 \\
&+ \frac{1}{2^9 \cdot 5^3} \begin{pmatrix} 52800x^{12} + 211440x^{11} + 414553x^{10} + 528350x^9 \\ + 484509x^8 + 337200x^7 + 182410x^6 + 77532x^5 \\ +25722x^4 + 6544x^3 + 1213x^2 + 150x + 9 \end{pmatrix} y \\
&- \frac{1}{2^{12} \cdot 5^4} \begin{pmatrix} 960000x^{16} + 3224000x^{15} + 5184200x^{14} + 4513325x^{13} \\ + 1189308x^{12} - 2464974x^{11} - 4252380x^{10} \\ - 3906469x^9 - 2536840x^8 - 1251364x^7 - 479056x^6 \\ - 142093x^5 - 31924x^4 - 5166x^3 - 540x^2 - 27x \end{pmatrix} = 0.
\end{aligned}$$

5.3.2. *Data for $X_0^+(197)$.* The equations for the canonical embedding in $\mathbf{P}^{g-1}$ are

$$\begin{aligned}
&WX - UY + VY = 0,\\
&VW + X^2 - UY - XY - XZ = 0,\\
&UX - UY + VY + WY + YZ = 0,\\
&U^2 - UV - UW + VW + W^2 + WZ = 0,\\
&V^2 - VX + VY + UZ - WZ + XZ - YZ = 0,\\
&U^2 - UW + W^2 - X^2 + VY + WY + XZ = 0,
\end{aligned}$$

with variables corresponding to the following cusp forms:

$$\begin{aligned}
U &= -q^3 + q^4 + q^5 - q^8 + q^9 - q^{10} + 2q^{11} - q^{13} - q^{14} + q^{15} - q^{16} - 4q^{17}\\
&\quad + 2q^{18} + 2q^{19} + O(q^{20}),\\
V &= -q^2 + q^4 + q^5 + 2q^6 + q^{11} - q^{12} - q^{13} + 2q^{14} - 2q^{15} - q^{16} - q^{17}\\
&\quad - 3q^{18} + O(q^{20}),\\
W &= q^4 - q^5 - q^6 - q^{10} + 3q^{11} - q^{12} + 3q^{15} - 2q^{16} - 2q^{17} + 4q^{18}\\
&\quad - q^{19} + O(q^{20}),\\
X &= -q^5 + q^6 + q^7 + q^9 - q^{10} - 2q^{12} - q^{13} + 4q^{17} - 4q^{18} - 2q^{19} + O(q^{20}),\\
Y &= -q^7 + q^8 + 2q^9 - 2q^{11} - 2q^{12} + 2q^{13} - q^{14} - q^{15} + 3q^{17} - q^{18}\\
&\quad - q^{19} + O(q^{20}),\\
Z &= q - q^3 - q^4 - q^5 - q^7 - q^8 - q^{10} - 2q^{11} - 3q^{13} + q^{14} - q^{15} + q^{16} + 2q^{17}\\
&\quad - 2q^{18} - 3q^{19} + O(q^{20}).
\end{aligned}$$

The eight known rational points are

Cusp, $[0:0:0:0:0:1]$,	$D=-19$, $[0:0:0:0:1:0]$,
$D=-4$, $[1:0:1:1:1:-1]$,	$D=-28$, $[1:1:0:1:1:-1]$,
$D=-7$, $[1:0:1:-1:-1:-1]$,	$D=-43$, $[0:0:1:0:-1:-1]$,
$D=-16$, $[1:1:0:-1:-1:-1]$,	$D=-163$, $[2:6:-2:2:1:-6]$.

For quadratic Chabauty, we use $p = 23$ and two models $\{(\varphi_{197,i}, \mathcal{C}_{197,i})\}_{i=1}^{2}$. Information for $\varphi_{197,1}$:

$$\begin{aligned}
&d_x = 6, \quad x_1 = U + Z, \qquad x_2 = 2X - 2Y,\\
&d_y = 6, \quad y_1 = V + W + Z, \quad y_2 = X - Y.
\end{aligned}$$

Information for $\varphi_{197,2}$:

$$\begin{aligned}
&d_x = 5, \quad x_1 = U + Z, \quad x_2 = V + W + Z,\\
&d_y = 6, \quad y_1 = X - Y, \quad y_2 = V + W + Z.
\end{aligned}$$

The equation of $\mathcal{C}_{197,1}$ in the affine patch where $z = 1$ is

$$\begin{aligned} &y^6 - 10xy^5 + (44x^2 + 10x - 3)y^4 + (-112x^3 - 48x^2 + 2x + 2)y^3 \\ &\quad + (176x^4 + 112x^3 + 16x^2 - 4x + 2)y^2 \\ &\quad + (-160x^5 - 128x^4 - 40x^3 - 2x - 2)y \\ &\quad + (64x^6 + 64x^5 + 32x^4 + 2x) = 0. \end{aligned}$$

The equation of $\mathcal{C}_{197,2}$ in the affine patch where $z = 1$ is

$$\begin{aligned} &y^5 + (-x + 2)y^4 + (-2x^2 + 6x - 4)y^3 \\ &\quad + (2x^6 - 13x^5 + 32x^4 - 35x^3 + 9x^2 + 16x - 12)y^2 \\ &\quad + (2x^8 - 20x^7 + 86x^6 - 208x^5 + 309x^4 - 286x^3 + 156x^2 - 40x)y \\ &\quad + \begin{pmatrix} x^{10} - 13x^9 + 75x^8 - 254x^7 + 563x^6 \\ - 861x^5 + 929x^4 - 704x^3 + 360x^2 - 112x + 16 \end{pmatrix} = 0. \end{aligned}$$

5.3.3. *Data for $X_0^+(211)$.* The equations for the canonical embedding in $\mathbf{P}^{g-1}$ are

$$\begin{aligned} &UV - V^2 + X^2 - UY + VY = 0, \\ &W^2 - VX - WX + WY - XY + YZ = 0, \\ &UV + VX + WX - UY - VZ + YZ = 0, \\ &VW + UX - VX - WX + X^2 - WY + YZ = 0, \\ &UW - VX - WX + UY - VY - WY + XZ + YZ = 0, \\ &U^2 - UW + UX - WX - VY - VZ + WZ + YZ = 0, \end{aligned}$$

with variables corresponding to the following cusp forms:

$$\begin{aligned} U &= -q^2 + q^3 + q^4 - q^7 - q^9 + 2q^{10} + q^{12} + 2q^{13} + 2q^{14} - 2q^{15} - q^{16} - q^{17} \\ &\quad + q^{19} + O(q^{20}), \\ V &= q^5 - 2q^7 - q^8 - q^9 + 2q^{11} + 2q^{12} + q^{13} + q^{14} - q^{18} + O(q^{20}), \\ W &= q^3 - q^4 - q^5 + q^8 - q^9 + q^{10} - q^{13} - 3q^{15} + q^{16} + 2q^{17} - q^{18} \\ &\quad + q^{19} + O(q^{20}), \\ X &= -q^4 + q^5 + q^6 - q^9 + q^{10} + q^{12} - 2q^{13} - q^{14} - 2q^{15} + 2q^{16} + q^{17} - 2q^{18} \\ &\quad - q^{19} + O(q^{20}), \\ Y &= q^5 - q^6 - q^7 - q^{11} + 2q^{12} + 3q^{14} - 3q^{17} + q^{18} + q^{19} + O(q^{20}), \\ Z &= -q + q^3 + 2q^5 + q^6 + q^8 + q^{10} + 2q^{11} + q^{12} + q^{13} + q^{14} - 2q^{15} + 2q^{17} \\ &\quad - 2q^{18} + q^{19} + O(q^{20}). \end{aligned}$$

The eight known rational points are

$$\begin{array}{ll} \text{Cusp, } [0:0:0:0:0:1], & D=-12, [1:1:0:0:0:1], \\ D=-3, [1:-3:2:2:-2:3], & D=-27, [1:0:-1:-1:1:0], \\ D=-7, [1:1:0:0:1:0], & D=-28, [1:-1:0:-2:1:0], \\ D=-8, [0:0:0:0:1:0], & D=-67, [1:0:1:1:1:0]. \end{array}$$

For quadratic Chabauty, we use $p=31$ and two models $\{(\varphi_{211,i}, \mathcal{C}_{211,1})\}_{i=1}^{2}$.
Information for $\varphi_{211,1}$:

$$\begin{array}{lll} d_x = 6, & x_1 = X - Y - Z, & x_2 = V - W - 2Y, \\ d_y = 6, & y_1 = U + 2Y + Z, & y_2 = V - W - 2Y. \end{array}$$

Information for $\varphi_{211,2}$:

$$\begin{array}{lll} d_x = 6, & x_1 = U + 2Y + Z, & x_2 = X - Y - Z, \\ d_y = 6, & y_1 = V - W - 2Y, & y_2 = X - Y - Z, \end{array}$$

followed by the automorphism $[-C, B+C, A+2C] : \mathbf{P}^2 \to \mathbf{P}^2$.

The equation of $\mathcal{C}_{211,1}$ in the affine patch where $z=1$ is

$$\begin{aligned} y^6 &+ (9x+1)y^5 + (32x^2+12x-2)y^4 + (57x^3+49x^2-6x-4)y^3 \\ &+ (53x^4+89x^3+5x^2-19x-1)y^2 \\ &+ (24x^5+72x^4+30x^3-26x^2-8x+2)y \\ &+ (4x^6+20x^5+24x^4-8x^3-14x^2+3x+1) = 0. \end{aligned}$$

The equation of $\mathcal{C}_{211,2}$ in the affine patch where $z=1$ is

$$\begin{aligned} y^6 &+ (7x+2)y^5 + (18x^2+14x-1)y^4 + (22x^3+38x^2-9x-4)y^3 \\ &+ (13x^4+48x^3-28x^2-22x-2)y^2 \\ &+ (3x^5+28x^4-36x^3-39x^2-6x+1)y \\ &+ (6x^5-17x^4-22x^3-2x^2+4x+1) = 0, \end{aligned}$$

with $\mathbf{F}_p$-points $[-1:1:1]$, $[-1/2:0:1]$, $[1:-2:1]$, $[-2:3:1]$, $[-1:2:1]$, $[0:-1:1]$ on the affine patch with $z=1$ and infinite points $[-1/3:1:0]$, $[1:0:0]$.

5.3.4. *Data for* $X_0^+(223)$. The equations for the canonical embedding in $\mathbf{P}^{g-1}$ are

$$\begin{aligned} &U^2+V^2-UW+VX=0, \\ &VX+VY-WY+XY+XZ+YZ=0, \\ &V^2+VW+VY+WY-UZ+VZ+WZ-YZ=0, \\ &V^2+UW-W^2-UX+WX-VY-UZ+WZ=0, \\ &UV-VW+VX-WX+X^2-UY+VZ+XZ=0, \\ &UV+V^2+UW-UX+WX-X^2+3UY-VY \\ &\qquad\qquad\qquad +UZ+VZ-WZ+Z^2=0, \end{aligned}$$

with variables corresponding to the following cusp forms:

$$
\begin{aligned}
U &= q^6 - q^7 - q^8 - q^{10} + 2q^{11} - q^{12} + q^{13} + q^{14} + 3q^{16} - 2q^{18} \\
&\quad - 2q^{19} + O(q^{20}), \\
V &= -q^5 + q^6 + q^7 - 2q^{12} + 2q^{13} - 2q^{14} - q^{16} + q^{17} - q^{18} - q^{19} + O(q^{20}), \\
W &= -q^2 + q^3 + q^4 - q^7 - q^9 + q^{10} + q^{11} + q^{12} + 2q^{14} - q^{15} + 3q^{16} + q^{17} \\
&\quad - 3q^{19} + O(q^{20}), \\
X &= q^3 - q^4 - q^6 - q^7 + 2q^8 - 2q^9 + 2q^{12} - q^{13} + q^{14} - q^{15} + 2q^{17} + 2q^{18} \\
&\quad - 2q^{19} + O(q^{20}), \\
Y &= -q + q^2 + q^5 + q^7 + 2q^9 - q^{10} + q^{11} + 2q^{13} + q^{15} - q^{16} + 2q^{17} - q^{18} \\
&\quad + q^{19} + O(q^{20}), \\
Z &= -q^2 + q^4 + q^5 + q^6 + q^8 + q^{10} + q^{11} - q^{12} - 2q^{13} - q^{19} + O(q^{20}).
\end{aligned}
$$

The seven known rational points are

$$
\begin{array}{ll}
\text{Cusp}, [0:0:0:0:1:0], & D = -27, [0:1:-1:-1:1:0], \\
D = -3, [3:-2:5:-1:-2:6], & D = -67, [0:1:1:-1:-1:0], \\
D = -11, [0:0:1:1:0:0], & D = -163, [2:-2:1:3:-4:-6]. \\
D = -12, [1:0:1:1:0:0], &
\end{array}
$$

For quadratic Chabauty, we use $p = 19$ on two patches of the same projective model $\{(\varphi_{223,i}, \mathcal{C}_{223,i})\}_{i=1}^{2}$.

Information for $\varphi_{223,1}$:

$$
\begin{aligned}
d_x &= 6, \quad x_1 = U - W, \qquad x_2 = X + Y, \\
d_y &= 5, \quad y_1 = 2(V - Y), \quad y_2 = X + Y.
\end{aligned}
$$

Information for $\varphi_{223,2}$:

$$
\begin{aligned}
d_x &= 6, \quad x_1 = X + Y, \qquad x_2 = U - W, \\
d_y &= 5, \quad y_1 = 2(V - Y), \quad y_2 = U - W.
\end{aligned}
$$

The equation of $\mathcal{C}_{223,1}$ in the affine patch where $z = 1$ is

$$
\begin{aligned}
y^6 &+ (-x + 8)y^5 + (8x^2 + 30)y^4 + (-4x^3 + 88x^2 + 40x + 60)y^3 \\
&+ (8x^4 - 32x^3 + 264x^2 + 128x + 64)y^2 \\
&+ (-16x^5 - 192x^3 + 256x^2 + 176x + 32)y \\
&+ (-96x^5 - 128x^4 - 160x^3 + 128x) = 0.
\end{aligned}
$$

The equation of $\mathcal{C}_{223,2}$ in the affine patch where $z = 1$ is

$$
\begin{aligned}
y^6 &+ (8x - 1)y^5 + (30x^2 + 8)y^4 + (60x^3 + 40x^2 + 88x - 4)y^3 \\
&+ (64x^4 + 128x^3 + 264x^2 - 32x + 8)y^2 \\
&+ (32x^5 + 176x^4 + 256x^3 - 192x^2 - 16)y \\
&+ (128x^5 - 160x^3 - 128x^2 - 96x) = 0.
\end{aligned}
$$

5.3.5. *Data for $X_0^+(269)$.* The equations for the canonical embedding in $\mathbf{P}^{g-1}$ are

$$\begin{aligned}
&UY - WZ + XZ = 0,\\
&WY - Y^2 + VZ - WZ = 0,\\
&UW + VW - VY - UZ - VZ - YZ = 0,\\
&UV - UW + W^2 - WX - WY + XY = 0,\\
&V^2 - VW - W^2 + WX + UY + VY + Y^2 = 0,\\
&U^2 - VW - W^2 + UX - VX + X^2 - XZ = 0,
\end{aligned}$$

with variables corresponding to the following cusp forms:

$$\begin{aligned}
U &= -q^2 + q^3 + q^6 + q^8 - q^9 + q^{10} + q^{11} - q^{13} + 2q^{14} - 2q^{15} + q^{16} - q^{17}\\
&\quad - q^{19} + O(q^{20}),\\
V &= -q^3 + q^5 + q^6 + q^7 + q^9 - q^{10} - q^{11} + q^{12} - q^{16} - 2q^{18} + q^{19} + O(q^{20}),\\
W &= -q^7 + q^8 + q^9 - q^{12} + 2q^{13} - 2q^{14} - q^{15} - q^{17} - q^{19} + O(q^{20}),\\
X &= -q^3 + q^4 + q^7 + 2q^9 - q^{10} - q^{14} - 3q^{16} + 2q^{17} - 2q^{19} + O(q^{20}),\\
Y &= q^2 - q^4 - q^5 - q^6 + q^9 - q^{11} - q^{12} - 2q^{14} + q^{16} - q^{17} - q^{18} + O(q^{20}),\\
Z &= -q + q^3 + q^4 + q^7 + q^{10} + 3q^{11} + q^{13} + q^{15} + q^{17} + q^{18} + 2q^{19} + O(q^{20}).
\end{aligned}$$

The six known rational points are

$$\begin{array}{ll}
\text{Cusp, } [0:0:0:0:0:1], & D=-16,\ [0:1:-1:1:-1:0],\\
D=-4,\ [0:1:1:-1:1:0], & D=-43,\ [1:-1:1:-1:0:0],\\
D=-11,\ [0:0:1:1:0:0], & D=-67,\ [1:-1:-1:-1:0:2].
\end{array}$$

For quadratic Chabauty, we use $p = 29$ and a single model $(\varphi_{269,1}, \mathcal{C}_{269,1})$. Information for $\varphi_{269,1}$:

$$\begin{array}{lll}
d_x = 4, & x_1 = 2Y - 2Z, & x_2 = 2Y - Z,\\
d_y = 6, & y_1 = U, & y_2 = W - Y + 2Z.
\end{array}$$

The equation of $\mathcal{C}_{269,1}$ in the affine patch where $z = 1$ is

$$\begin{aligned}
y^4 &+ \frac{1}{3\cdot 11}\left(121x^6 - 1045x^5 + 3614x^4 - 6388x^3 + 6048x^2 - 2768x + 320\right)y^3\\
&+ \frac{1}{3^2\cdot 11^2}\begin{pmatrix} 2145x^{12} - 54444x^{11} + 276793x^{10} - 630098x^9 + 790076x^8\\ - 845272x^7 + 1657072x^6 - 3337920x^5 + 4271616x^4\\ - 3294464x^3 + 1493248x^2 - 378880x + 53248 \end{pmatrix} y^2\\
&+ \frac{1}{3^3\cdot 11^3}\begin{pmatrix} -47916x^{18} + 2646270x^{17} - 54539958x^{16} + 557744070x^{15}\\ - 3479080646x^{14} + 14677622884x^{13} - 44393492640x^{12}\\ + 99794532256x^{11} - 170613685632x^{10} + 225022856064x^9\\ - 230718110592x^8 + 184336004352x^7 - 114511136768x^6\\ + 54945251328x^5 - 20124897280x^4 + 5510791168x^3\\ - 1082523648x^2 + 139460608x - 9437184 \end{pmatrix} y
\end{aligned}$$

$$+\frac{1}{3^4\cdot 11^4}\left(\begin{array}{c} -1006236x^{24}+76591548x^{23}-2333162502x^{22} \\ +37125740826x^{21}-348516546390x^{20}+2155553249634x^{19} \\ -9476259572816x^{18}+31140876323216x^{17}-79133128205312x^{16} \\ +159133059208064x^{15}-257270458757504x^{14} \\ +337929123632384x^{13}-363041352011776x^{12} \\ +320161247417344x^{11}-232066303320064x^{10} \\ +138148922114048x^{9}-67352475975680x^{8}+26747617058816x^{7} \\ -8571322695680x^{6}+2180924702720x^{5}-428701908992x^{4} \\ +62014881792x^{3}-5989466112x^{2}+301989888x \end{array}\right)=0.$$

5.3.6. *Data for* $X_0^+(271)$. The equations for the canonical embedding in $\mathbf{P}^{g-1}$ are

$$\begin{aligned} &UV+VX-WX-WY+VZ=0,\\ &U^2+UV-UW-UY-XY-Z^2=0,\\ &UV+V^2+UX+UY+XY+Y^2+VZ+2WZ=0,\\ &U^2-VX-XY-Y^2+UZ-VZ+XZ+YZ=0,\\ &VW-VX+UY+VY+XY+Y^2+UZ+XZ+YZ=0,\\ &UW-VW-UX-VX-UY-2VY-Y^2-VZ-WZ+XZ=0, \end{aligned}$$

with variables corresponding to the following cusp forms:

$$\begin{aligned} U&=-q^3+q^5+q^6+q^8+q^9+q^{11}-q^{12}-2q^{14}+q^{15}-3q^{16}-q^{17}\\ &\quad -q^{18}+O(q^{20}),\\ V&=q^4-q^5-2q^8+q^{11}+2q^{13}-q^{14}+q^{19}+O(q^{20}),\\ W&=-q+q^2+q^5+q^9-q^{10}+3q^{11}+q^{13}+q^{15}-q^{16}+2q^{17}+q^{19}+O(q^{20}),\\ X&=q^2-q^4-q^5-q^8-q^9-2q^{10}+q^{11}+q^{13}-q^{18}+O(q^{20}),\\ Y&=-q^2+q^4+q^6+q^7+q^8+q^9+q^{10}-q^{11}-q^{12}+q^{13}-q^{16}-q^{17}\\ &\quad +q^{19}+O(q^{20}),\\ Z&=q^7-q^8-q^9+q^{12}-q^{13}+3q^{16}+q^{17}+q^{18}-2q^{19}+O(q^{20}) \end{aligned}$$

The six known rational points are

$$\begin{array}{ll} \text{Cusp, } [0:0:1:0:0:0], & D=-19,\ [0:0:0:1:0:0],\\ D=-3,\ [1:-4:-2:-3:5:3], & D=-27,\ [1:-1:1:0:-1:0],\\ D=-12,\ [1:0:0:-1:1:-1], & D=-43,\ [0:0:0:1:-1:1]. \end{array}$$

For quadratic Chabauty, we use $p=13$ and a single model $(\varphi_{271,1},\mathcal{C}_{271,1})$. Information for $\varphi_{271,1}$:

$$\begin{array}{lll} d_x=5, & x_1=V+W, & x_2=U-W+2X+Y,\\ d_y=9, & y_1=80(2W-2X-Y+Z), & y_2=U-W+2X+Y. \end{array}$$

The equation of $\mathcal{C}_{271,1}$ in the affine patch where $z = 1$ is

$$\begin{aligned}
&y^5 + \big(-85x^5 + 1127x^4 - 1913x^3 - 1182x^2 - 163x - 6\big)y^4 \\
&\quad + \begin{pmatrix} 10080x^{10} - 140412x^9 + 690654x^8 - 1381518x^7 + 483107x^6 \\ + 1493128x^5 + 629335x^4 + 89925x^3 + 672x^2 - 820x - 48 \end{pmatrix} y^3 \\
&\quad + \begin{pmatrix} 51200x^{15} + 5322240x^{14} - 62218208x^{13} + 256115488x^{12} \\ - 497719884x^{11} + 393876380x^{10} + 397941188x^9 - 406993184x^8 \\ - 419868463x^7 - 107094101x^6 + 10069328x^5 + 10921601x^4 \\ + 2495279x^3 + 280582x^2 + 16084x + 376 \end{pmatrix} y^2 \\
&\quad + \begin{pmatrix} -77824000x^{20} + 842444800x^{19} - 1778347520x^{18} - 10376567744x^{17} \\ + 57185179088x^{16} - 105874923384x^{15} + 82483583448x^{14} + 37040438788x^{13} \\ - 128890371748x^{12} - 9768338538x^{11} + 103912556718x^{10} + 65061175451x^9 \\ + 8351258644x^8 - 6617560964x^7 - 3771233527x^6 - 977933013x^5 \\ - 153099583x^4 - 15265548x^3 - 951824x^2 - 33920x - 528 \end{pmatrix} y \\
&\quad + \begin{pmatrix} 1966080000x^{25} - 32800768000x^{24} + 195656908800x^{23} - 384548372480x^{22} \\ - 835604847872x^{21} + 5438540280384x^{20} - 10251070348080x^{19} \\ + 7222065612944x^{18} + 4271687117728x^{17} - 10456733694624x^{16} \\ + 2541993008328x^{15} + 7501013423496x^{14} - 4899818383384x^{13} \\ - 10352281169140x^{12} - 5911510862587x^{11} - 1394541574295x^{10} \\ + 91020107428x^9 + 165638010722x^8 + 56008509377x^7 + 10935753977x^6 \\ + 1399983446x^5 + 119863400x^4 + 6648960x^3 + 216912x^2 + 3168x \end{pmatrix} = 0.
\end{aligned}$$

5.3.7. *Data for* $X_0^+(359)$. The equations for the canonical embedding in $\mathbf{P}^{g-1}$ are

$$\begin{aligned}
&UV + VX + VY + UZ + VZ + WZ = 0, \\
&UV + 2UW + VW + W^2 + WY + WZ + XZ = 0, \\
&U^2 - V^2 - VW - UX + UY + WY + Y^2 = 0, \\
&V^2 + VW + UY + VY + UZ + VZ - XZ + YZ = 0, \\
&U^2 + UW - UX - WX - WY + UZ + WZ - XZ = 0, \\
&UV + UW + VW - VX - WX - UY - WY - Y^2 - YZ = 0,
\end{aligned}$$

with variables corresponding to the following cusp forms:

$$\begin{aligned}
U &= -q^3 + q^5 + q^7 + q^9 + q^{10} + q^{12} - q^{13} - q^{16} - q^{17} + O(q^{20}), \\
V &= q^8 - q^9 - q^{10} + q^{13} + q^{15} - q^{16} + q^{19} + O(q^{20}), \\
W &= q^3 - q^6 - q^7 - q^8 - q^9 - q^{11} - q^{13} + q^{14} - q^{15} + 2q^{16} + q^{17} + 2q^{18} \\
&\quad + q^{19} + O(q^{20}), \\
X &= -q + q^4 + q^5 + q^7 + q^9 + q^{10} + q^{11} + q^{12} + q^{13} + q^{15} + 2q^{17} \\
&\quad + 2q^{19} + O(q^{20}),
\end{aligned}$$

$$Y = q^2 - q^4 - q^5 - q^{10} - q^{12} + q^{13} - q^{14} - q^{17} - 2q^{18} + O(q^{20}),$$
$$Z = q^4 - q^5 - q^8 - q^{10} + q^{11} + q^{15} - q^{16} + 2q^{17} - q^{18} - q^{19} + O(q^{20}).$$

The seven known rational points are

$$\begin{array}{ll} \text{Cusp}, [0:0:0:1:0:0], & D=-43, [1:0:0:1:0:0], \\ D=-7, [0:1:0:0:1:-1], & D=-67, [1:0:-2:1:0:0], \\ D=-19, [0:0:0:0:0:1], & D=-163, [0:2:-6:6:4:5], \\ D=-28, [0:1:0:0:-1:1], & \end{array}$$

For quadratic Chabauty, we use $p = 7$ and two patches of the same projective model $\{(\varphi_{359,i}, \mathcal{C}_{359,i})\}_{i=1}^{2}$.

Information for $\varphi_{359,1}$:

$$d_x = 6, \quad x_1 = W + Y, \quad x_2 = X,$$
$$d_y = 6, \quad y_1 = U + V, \quad y_2 = X.$$

Information for $\varphi_{359,2}$:

$$d_x = 6, \quad x_1 = X, \quad x_2 = U + V,$$
$$d_y = 5, \quad y_1 = W + Y, \quad y_2 = U + V.$$

The equation of $\mathcal{C}_{359,1}$ in the affine patch where $z = 1$ is

$$\begin{aligned} y^6 &+ (2x+1)y^5 + (4x^2 - 10x - 6)y^4 + (2x^3 - 15x^2 - 7x + 9)y^3 \\ &+ (-4x^4 - 10x^3 - 9x^2 + 12x - 6)y^2 \\ &+ (-4x^5 - 4x^4 - 6x^3 + 8x^2 - 3x + 1)y \\ &+ (-x^6 - x^5 - 2x^4 + x^3 - x^2) = 0. \end{aligned}$$

The equation of $\mathcal{C}_{359,2}$ in the affine patch where $z = 1$ is

$$\begin{aligned} y^6 &+ (x+4)y^5 + (2x^2 + 4x + 4)y^4 + (-x^3 + 6x^2 + 10x - 2)y^3 \\ &+ (x^4 - 8x^3 + 9x^2 + 15x - 4)y^2 + (3x^4 - 12x^3 + 7x^2 + 10x - 2)y \\ &+ (-x^5 + 6x^4 - 9x^3 + 6x^2 - x - 1) = 0. \end{aligned}$$

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References

[AAB+21] N. Adžaga, V. Arul, L. Beneish, M. Chen, S. Chidambaram, T. Keller, and B. Wen, *Source code for this article*, https://github.com/AWS2020QC/ModularCurvesX0plusG4-6, 2021.

[ACKP22] N. Adžaga, S. Chidambaram, T. Keller, and O. Padurariu, *Rational points on hyperelliptic Atkin–Lehner quotients of modular curves and their coverings*, Res. Number Theory 8 (2022), no. 4, art. 87, 24 pp.

[AM10] K. Arai and F. Momose, *Rational points on* $X_0^+(37M)$, J. Number Theory 130 (2010), 2272–2282.

[AM22] V. Arul and J. S. Müller, *Rational points on* $X_0^+(125)$, arXiv:2205.14744, (2022).

[BH03] M. Baker and Y. Hasegawa, *Automorphisms of* $X_0^*(p)$, J. Number Theory 100 (2003), 72–87.

[BBB+20] J. S. Balakrishnan, A. J. Best, F. Bianchi, N. Dogra, B. Lawrence, J. S. Müller, N. Triantafillou, J. Tuitman, and J. Vonk, *QCMod*, https://github.com/steffenmueller/QCMod, 2020.

[BBB+21] J. S. Balakrishnan, A. J. Best, F. Bianchi, B. Lawrence, J. S. Müller, N. Triantafillou, and J. Vonk, *Two recent p-adic approaches towards the (effective) Mordell conjecture*, in: Arithmetic L-Functions and Differential Geometric Methods, Progr. Math. 338, Birkhäuser/Springer, Cham, 2021, 37–74.

[BD18] J. S. Balakrishnan and N. Dogra, *Quadratic Chabauty and rational points, I: p-adic heights*, Duke Math. J. 167 (2018), 1981–2038.

[BD19] J. S. Balakrishnan and N. Dogra, *An effective Chabauty–Kim theorem*, Compos. Math. 155 (2019), 1057–1075.

[BD21] J. S. Balakrishnan and N. Dogra, *Quadratic Chabauty and rational points II: Generalised height functions on Selmer varieties*, Int. Math. Res. Notices 2021, 11923–12008.

[BDM+19] J. S. Balakrishnan, N. Dogra, J. S. Müller, J. Tuitman, and J. Vonk, *Explicit Chabauty–Kim for the split Cartan modular curve of level 13*, Ann. of Math. (2) 189 (2019), 885–944.

[BDM+21] J. S. Balakrishnan, N. Dogra, J. S. Müller, J. Tuitman, and J. Vonk, *Quadratic Chabauty for modular curves: algorithms and examples*, arXiv:2101.01862v2 (2021).

[BM20] J. S Balakrishnan and J. S. Müller, *Computational tools for quadratic Chabauty*, preprint, Boston Univ., 2020.

[BT20] J. S. Balakrishnan and J. Tuitman, *Explicit Coleman integration for curves*, Math. Comp. 89 (2020), no. 326, 2965–2984.

[BG21] F. Bars and J. González, *The automorphism group of the modular curve* $X_0^*(N)$ *with square-free level*, Trans. Amer. Math. Soc. 374 (2021), 5783–5803.

[BGX21] F. Bars, J. González, and X. Xarles, *Hyperelliptic parametrizations of* $\mathbb{Q}$*-curves*, Ramanujan J. 56 (2021), 103–120.

[BCP97] W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. 24 (1997), 235–265.

[CB09] C. Castaño-Bernard, *A note on the rational points of $X_0^+(N)$*, Experiment. Math. 18 (2009), 129–135.

[Cha41] C. Chabauty, *Sur les points rationnels des courbes algébriques de genre supérieur à l'unité*, C. R. Acad. Sci. Paris 212 (1941), 882–885.

[Cla03] P. L. Clark, *Rational points on Atkin–Lehner quotients of Shimura curves*, Ph.D. thesis, Harvard Univ., 2003.

[Col85] R. F. Coleman, *Effective Chabauty*, Duke Math. J. 52 (1985), 765–770.

[CWZ00] J. A. Csirik, J. L. Wetherell and M. E. Zieve, *On the genera of $X_0(N)$*, arXiv:math/0006096.

[DS05] F. Diamond and J. Shurman, *A First Course in Modular Forms*, Grad. Texts in Math. 228, Springer, New York, 2005.

[DLF21] N. Dogra and S. Le Fourn, *Quadratic Chabauty for modular curves and modular forms of rank one*, Math. Ann. 380 (2021), 393–448.

[EL23] B. Edixhoven and G. Lido, *Geometric quadratic Chabauty*, J. Just. Math. Jussien 22 (2023), 279–333.

[FH99] M. Furumoto and Y. Hasegawa, *Hyperelliptic quotients of modular curves $X_0(N)$*, Tokyo J. Math. 22 (1999), 105–125.

[Gal96] S. D. Galbraith, *Equations for modular curves*, DPhil thesis, Univ. of Oxford, 1996.

[Gal99] S. D. Galbraith, *Rational points on $X_0^+(p)$*, Experiment. Math. 8 (1999), 311–318.

[Gal02] S. D. Galbraith, *Rational points on $X_0^+(N)$ and quadratic $\mathbb{Q}$-curves*, J. Théor. Nombres Bordeaux 14 (2002), 205–219.

[Gro84] B. H. Gross, *Heegner points on $X_0(N)$*, in: Modular Forms (Durham, 1983), Ellis Horwood Ser. Math. Appl. Statist. Oper. Res., Ellis Horwood, Chichester, 1984, 87–105.

[GZ86] B. H. Gross and D. B. Zagier, *Heegner points and derivatives of L-series*, Invent. Math. 84 (1986), 225–320.

[Har77] R. Hartshorne, *Algebraic Geometry*, Grad. Texts in Math. 52, Springer, New York, 1977.

[HH96] Y. Hasegawa and K. Hashimoto, *Hyperelliptic modular curves $X_0^*(N)$ with square-free levels*, Acta Arith. 77 (1996), 179–193.

[Kim05] M. Kim, *The motivic fundamental group of $\mathbf{P}^1 \setminus \{0, 1, \infty\}$ and the theorem of Siegel*, Invent. Math. 161 (2005), 629–656.

[Kim09] M. Kim, *The unipotent Albanese map and Selmer varieties for curves*, Publ. Res. Inst. Math. Sci. 45 (2009), 89–133.

[KL89] V. A. Kolyvagin and D. Yu. Logachëv, *Finiteness of the Shafarevich–Tate group and the group of rational points for some modular abelian varieties*, Algebra i Analiz 1 (1989), no. 5, 171–196 (in Russian).

[Lou11] S. R. Louboutin, *Upper bounds for residues of Dedekind zeta functions and class numbers of cubic and quartic number fields*, Math. Comp. 80 (2011), no. 275, 1813–1822.

[MP12] W. McCallum and B. Poonen, *The method of Chabauty and Coleman*, in: Explicit Methods in Number Theory, Panor. Synthèses 36, Soc. Math. France, Paris, 2012, 99–117.

[Mer18] P. Mercuri, *Equations and rational points of the modular curves $X_0^+(p)$*, Ramanujan J. 47 (2018), 291–308.

[Mom86] F. Momose, *Rational points on the modular curves $X_0^+(p^r)$*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 33 (1986), 441–466.

[Mom87] F. Momose, *Rational points on the modular curves $X_0^+(n)$*, J. Math. Soc. Japan 39 (1987), 269–286.

[Nek93] J. Nekovář, *On p-adic height pairings*, in: Séminaire de Théorie des Nombres, Paris, 1990–91, Progr. Math. 108, Birkhäuser, Boston, 1993, 127–202.

[PY07] P. Parent and A. Yafaev, *Proving the triviality of rational points on Atkin–Lehner quotients of Shimura curves*, Math. Ann. 339 (2007), 915–935.

[Sik17] S. Siksek, *Quadratic Chabauty for modular curves*, arXiv:1704.00473 (2017).

[Sto19] M. Stoll, *Uniform bounds for the number of rational points on hyperelliptic curves of small Mordell–Weil rank*, J. Eur. Math. Soc. 21 (2019), 923–956.

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Abstract (will appear on the journal's web site only)

We use the method of quadratic Chabauty on the quotients $X_0^+(N)$ of modular curves $X_0(N)$ by their Fricke involutions to provably compute all the rational points of these curves for prime levels N of genus 4, 5, and 6. We find that the only such curves with exceptional rational points are of levels 137 and 311. In particular there are no exceptional rational points on those curves of genus 5 and 6. More precisely, we determine the rational points on the curves $X_0^+(N)$ for $N = 137, 173, 199, 251, 311, 157, 181, 227, 263, 163, 197, 211, 223, 269, 271, 359$.