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Source / Izvornik: **Nagoya mathematical journal, 2022, 246, 430 - 470**

Journal article, Published version

Rad u časopisu, Objavljena verzija rada (izdavačev PDF)

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LIE ALGEBRA MODULES WHICH ARE LOCALLY FINITE OVER THE SEMI-SIMPLE PART

VOLODYMYR MAZORCHUK AND RAFAEL MRĐEN

ABSTRACT. For a finite-dimensional Lie algebra $\mathfrak{L}$ over $\mathbb{C}$ with a fixed Levi decomposition $\mathfrak{L} = \mathfrak{g} \ltimes \mathfrak{r}$ where $\mathfrak{g}$ is semi-simple, we investigate $\mathfrak{L}$ -modules which decompose, as $\mathfrak{g}$ -modules, into a direct sum of simple finite-dimensional $\mathfrak{g}$ -modules with finite multiplicities. We call such modules $\mathfrak{g}$ -Harish-Chandra modules. We give a complete classification of simple $\mathfrak{g}$ -Harish-Chandra modules for the Takiff Lie algebra associated to $\mathfrak{g} = \mathfrak{sl}_2$, and for the Schrödinger Lie algebra, and obtain some partial results in other cases. An adapted version of Enright's and Arkhipov's completion functors plays a crucial role in our arguments. Moreover, we calculate the first extension groups of infinite-dimensional simple $\mathfrak{g}$ -Harish-Chandra modules and their annihilators in the universal enveloping algebra, for the Takiff $\mathfrak{sl}_2$ and the Schrödinger Lie algebra. In the general case, we give a sufficient condition for the existence of infinite-dimensional simple $\mathfrak{g}$ -Harish-Chandra modules.

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1. INTRODUCTION AND DESCRIPTION OF THE RESULTS

Construction and classification of modules are two fundamental problems in representation theory. In most of the cases, the problem of classification of all modules is known to be very hard (i.e. wild), so one is naturally forced to consider special classes of modules, for example, simple modules. Classification of simple modules is also quite hard in most of the cases. For example, for complex semi-simple Lie algebras, classification of all simple modules is only known, in some sense, for the algebra $\mathfrak{sl}_2$, see [Blo]. At the same time, numerous families of simple modules for semi-simple Lie algebras are very well understood, for example, simple highest weight modules, see [Ve, Hum08], Whittaker modules, see [Kos78, BM], weight modules with finite-dimensional weight spaces, see [Mat00], and Gelfand-Zeitlin modules, see [DOF, EMV, Web] and references therein.

There are two natural generalizations of semi-simple Lie algebras: semi-simple Lie superalgebras and non-semi-simple Lie algebras. For semi-simple Lie superalgebras, a significant progress in classification of simple modules was recently made in [CM, CCM]. Basically, for a large class of Lie superalgebras, the results of [CM, CCM] reduce the problem of classification of simple modules to a similar problem for the even part of the superalgebra, which is a reductive Lie algebra. In contrast to this situation, for non-semi-simple Lie algebras, very little is known. Apart from the main result of [Blo], which, in addition to $\mathfrak{sl}_2$, classifies simple modules over the Borel subalgebra of $\mathfrak{sl}_2$, several

special classes of simple modules were studied for various specific non-semi-simple Lie algebras, see e.g. [DLMZ14, Wil11, CCS14, CC17, Lau18, BL17, BL18, MS19] and references therein. We will now look at some of these and some other results in more detail.

It seems that the so called current Lie algebras are the ones which are most studied and best understood. These are defined as tensor product of a reductive Lie algebra with a commutative unital associative algebra. For current Lie algebras, there is a full classification of simple weight modules with finite-dimensional weight spaces, see [Lau18]. Also, the highest weight theory for the truncated polynomial version of these Lie algebras is developed in [Wil11]. Moreover, the center of the universal enveloping algebras of such Lie algebras whose semi-simple part is of type A is described explicitly in [Mol].

A special family of truncated current Lie algebras is formed by the so called Takiff Lie algebras, studied originally in [Tak71], which correspond to the case when one tensors a reductive Lie algebra with the associative algebra of dual numbers. The Takiff $\mathfrak{sl}_2$ is also known as the complexification of the Lie algebra of the Euclidean group $E(3)$, the Lie group of orientation-preserving isometries of the three-dimensional Euclidean space. It belongs to the family of conformal Galilei algebras, see e.g. [LMZ14]. Category $\mathcal{O}$ for Takiff $\mathfrak{sl}_2$ was recently studied in [MS19] and simple weight modules were classified in [BL17].

The Schrödinger Lie algebra, see Section 5, is also an important and intensively studied example of a non-reductive Lie algebra. Its category $\mathcal{O}$ was studied in detail in [DLMZ14], lowest weight modules were classified in [DDM97], and simple weight modules were classified in [Dub14, BL18].

A slight modification of the Schrödinger Lie algebra, called the centerless Schrödinger Lie algebra belongs to the family of conformal Galilei algebras, see Section 7. As their names suggest, the Schrödinger Lie algebra and conformal Galilei algebras are of great importance in theoretical physics and seem to have originated from there. For example, the Schrödinger Lie algebra comes from the Schrödinger Lie group, the group of symmetries of the free particle Schrödinger equation, see [DDM97, Per77]. Conformal Galilei algebras are related to the non-relativistic version of the AdS/CFT correspondence, see [BG09].

Several papers studied a generalization of Whittaker modules (originally defined in [Kos78] for semi-simple Lie algebras), in the setup of conformal Galilei algebras and the Schrödinger Lie algebra, see [CCS14, CSZ16, LMZ14, CC17]. Quasi-Whittaker modules are modules on which the radical of the Lie algebra acts locally finitely.

In the present paper we initiate the study of modules over (non-semi-simple) Lie algebras on which the action of the semi-simple part of the Lie algebra is locally finite, that is, which are locally finite over the semi-simple part. This condition is, in a sense, the opposite to the condition defining quasi-Whittaker modules. The obvious examples of modules that are locally finite over the semi-simple part are simple finite-dimensional modules over the semi-simple part on which the radical of our Lie algebra acts trivially. However, we observe that, for many Lie algebras, there exist simple infinite-dimensional modules that are locally finite over the semi-simple part. This motivates the problem of classification of such modules, and we show that this problem can be completely answered for the Takiff Lie algebra of $\mathfrak{sl}_2$ and for the Schrödinger Lie algebra. Moreover, the answer is both non-trivial and interesting. To our best knowledge, such modules have not been studied before in the general case (however, for the Schrödinger Lie algebra and the Takiff $\mathfrak{sl}_2$, they belong to a larger family of weight modules studied

in detail in [BL17, BL18]). Let us now describe the content of the paper in more detail.

If $\mathfrak{L}$ is any finite-dimensional Lie algebra and $\mathfrak{g} \subseteq \mathfrak{L}$ its semi-simple Levi subalgebra, we study $\mathfrak{L}$ -modules whose restriction to $\mathfrak{g}$ decomposes into a direct sum of simple finite-dimensional $\mathfrak{g}$ -modules with finite multiplicities, and call them *$\mathfrak{g}$ -Harish-Chandra modules*. To justify the name, we note that there is an obvious analogy with the classical theory of $(\mathfrak{g}, K)$ -modules as in [Vog81], coming from the setup of real reductive Lie groups. In the classical theory, any $(\mathfrak{g}, K)$ -module splits as a direct sum of finite-dimensional modules over the compact group K , and moreover, the multiplicities are finite if the corresponding group representation is irreducible and unitary (a result by Harish-Chandra). In our setup there is no such automatic splitting, so we “pretend” that $\mathfrak{g}$ is compact, i.e., we consider only those $\mathfrak{L}$ -modules that split as $\mathfrak{g}$ -modules into a direct sum of finite-dimensional $\mathfrak{g}$ -modules with finite multiplicities. Hopefully, this analogy could be used to transfer parts of the Langlands classification, or the theory of minimal K -types into our non-reductive algebraic setup. There is another analogy of our setup with integrable modules over a Kac-Moody algebra, see [Kac90].

In Section 2, we introduce the basic setup that we work in. In Section 3, we roughly describe “universal” $\mathfrak{g}$ -Harish-Chandra modules for Takiff Lie algebras. In particular, we show that such Lie algebras do indeed always have simple infinite-dimensional $\mathfrak{g}$ -Harish-Chandra modules, see Corollary 8.

In Sections 4 and 5 we prove our most concrete results: Theorem 30 provides a complete classification of simple $\mathfrak{g}$ -Harish-Chandra modules for the Takiff $\mathfrak{sl}_2$, and Theorem 53 gives such a classification for the Schrödinger Lie algebra. These two answers have both clear similarities and differences. In both cases we crucially use the highest weight theory for corresponding algebras and appropriate analogues of completions functors. Also, in both cases, we can consider semi-simple $\mathfrak{g}$ -Harish-Chandra modules as a monoidal representation of the monoidal category of finite-dimensional $\mathfrak{sl}_2$ -modules. We found it surprising that the combinatorial properties of the corresponding monoidal representation in the Takiff $\mathfrak{sl}_2$ and the Schrödinger cases are rather different.

In case of the Takiff $\mathfrak{sl}_2$, we obtain a family of modules $V(n, \chi)$ which are naturally parameterized by $n \in \mathbb{Z}$ and $\chi \in \mathbb{C} \setminus \{0\}$. However, we show that this family has a redundancy via non-trivial isomorphisms $V(n, \chi) \cong V(-n, -\chi)$. Roughly speaking, $|n|$ is the minimal $\mathfrak{g}$ -type, and χ^2 is the “purely radical part” of the central character. This classifies all simple infinite-dimensional $\mathfrak{g}$ -Harish-Chandra modules.

In case of the Schrödinger Lie algebra, we obtain a similar family of modules $V(n, \chi)$ parameterized by $n \in \mathbb{Z}_{\geq 0}$ and $\chi \in \mathbb{C} \setminus \{0\}$. However, in contrast to the Takiff case, this family is irredundant.

The modules mentioned above are very explicitly described. In both cases, we, moreover, show that all groups of first self extensions of these modules are one-dimensional, see Theorems 31 and 55. Additionally, we prove that the annihilators of all the above modules in the universal enveloping algebra are centrally generated, see Theorems 37 and Corollary 57. Classification results in Sections 4 and 5 are deducible (with non-trivial effort) from more general results of [BL17, BL18], however, we provide a completely different, less computational and more conceptual approach.

For comparison, it is easy too see that the centerless Schrödinger Lie algebra does not admit simple infinite-dimensional $\mathfrak{g}$ -Harish-Chandra modules. Roughly speaking, because its purely radical part of the center is trivial, see Remark 54 for details.

We would like to point out that the methods we utilize for our classification go far beyond direct calculations. We use various functorial constructions, which include, in particular, an appropriate adjustment of Enright's completion functor (based on Arkhipov's twisting functor), [Enr79, Deo80, KM02, AS03, Ark04, KM05]. Further development of both, highest weight theory and properties of various Lie theoretic functors as in [MSt07], for non-semi-simple Lie algebras, should provide an opportunity for generalization of the results of this paper to, in the first step, other Takiff Lie algebras and, further, general finite-dimensional Lie algebras.

In the most general case of an arbitrary finite-dimensional complex Lie algebra $\mathfrak{L}$ and a non-trivial Levi subalgebra $\mathfrak{g}$, it is clear that simple $\mathfrak{g}$ -Harish-Chandra modules always exist. Namely, the finite-dimensional $\mathfrak{L}$ -modules are, of course, $\mathfrak{g}$ -Harish-Chandra modules. In Theorem 60 of Section 6 we give a general sufficient condition for existence of infinite-dimensional simple $\mathfrak{g}$ -Harish-Chandra modules. The sufficient condition, as we formulate it, requires that the nilradical of $\mathfrak{L}$ intersects the centralizer in $\mathfrak{L}$ of the Cartan subalgebra of $\mathfrak{g}$. In this case we manage to use highest weight theory for $\mathfrak{L}$, combined with various versions of twisting functors, to construct infinite-dimensional simple $\mathfrak{g}$ -Harish-Chandra modules. We also provide an example showing that our sufficient condition is not necessary, in general: the semi-direct product of $\mathfrak{sl}_2$ and its 4-dimensional simple module does not satisfy our sufficient condition and has trivial highest weight theory in the sense that its simple highest weight modules coincide with simple highest weight $\mathfrak{sl}_2$ -modules. However, using various combinatorial tricks from [HHLS18], we show that this Lie algebra does admit simple infinite-dimensional $\mathfrak{g}$ -Harish-Chandra modules. This result can be found in Subsection 7.2.

Finally, in Subsection 7.3, in particular Theorem 67, we classify a class of $\mathfrak{sl}_2$ -Harish-Chandra modules that are connected to highest weight modules, for the semi-direct product of $\mathfrak{sl}_2$ with its simple 5-dimensional module. The corresponding category of semi-simple $\mathfrak{g}$ -Harish-Chandra modules is, again, a monoidal representation of the monoidal category of finite-dimensional $\mathfrak{sl}_2$ -modules. But the combinatorics of this monoidal representation is completely different from the ones which we get in the Takiff and the Schrödinger cases, see Remark 74. In particular, contrary to the previous cases, in this case we obtain an example of two simple $\mathfrak{g}$ -Harish-Chandra modules with different sets of $\mathfrak{g}$ -types, but with the same minimal $\mathfrak{g}$ -type.

Comparison of the results of [Han19] with Lemmata 10 and 40 suggests a possibility of an interesting connection between $\mathfrak{g}$ -Harish-Chandra modules and higher-spin algebras from [PRS90].

Acknowledgments. This research was partially supported by the Swedish Research Council, Göran Gustafsson Stiftelse and Vergstiftelsen. R. M. was also partially supported by the QuantiXLie Center of Excellence grant no. KK.01.1.1.01.0004 funded by the European Regional Development Fund.

2. NOTATION AND PRELIMINARIES

We work over the complex numbers $\mathbb{C}$. For a Lie algebra $\mathfrak{a}$, we denote by $U(\mathfrak{a})$ the universal enveloping algebra of $\mathfrak{a}$.

Fix a finite-dimensional Lie algebra $\mathfrak{L}$ over $\mathbb{C}$, and fix its Levi decomposition $\mathfrak{L} \cong \mathfrak{g} \ltimes \mathfrak{r}$. This is a semi-direct product, where $\mathfrak{g}$ is a maximal semi-simple Lie subalgebra, unique up to conjugation, and $\mathfrak{r} = \text{Rad } \mathfrak{L}$ is the radical of $\mathfrak{L}$, i.e., the unique maximal solvable ideal.

Definition 1. An $\mathfrak{L}$ -module V is called $\mathfrak{g}$ -Harish-Chandra module, if the restriction of V to $\mathfrak{g}$ decomposes as a direct sum of simple finite-dimensional $\mathfrak{g}$ -modules, and moreover, each isomorphism class of simple finite-dimensional $\mathfrak{g}$ -modules occurs with a finite multiplicity in V .

A simple $\mathfrak{g}$ -submodule of a $\mathfrak{g}$ -Harish-Chandra module V is called a $\mathfrak{g}$ -type of V . The sum of all $\mathfrak{g}$ -submodules of V isomorphic to a given $\mathfrak{g}$ -type is called the $\mathfrak{g}$ -isotypic component of V determined by this $\mathfrak{g}$ -type.

Fix a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$. Every $\mathfrak{g}$ -Harish-Chandra module is a weight module with respect to $\mathfrak{h}$. However, infinite-dimensional $\mathfrak{g}$ -Harish-Chandra modules might have infinite-dimensional weight spaces.

Remark 2. Note that the notion of a $\mathfrak{g}$ -Harish-Chandra module is different from the notion of Harish-Chandra module from [Lau18]. In the latter paper, Harish-Chandra modules are weight modules with finite-dimensional weight spaces. It would be natural to call the modules from [Lau18] $\mathfrak{h}$ -Harish-Chandra modules.

Denote by $\text{Nrad}(\mathfrak{L})$ the *nilradical* of $\mathfrak{L}$, by which we mean the intersection of kernels of all finite-dimensional simple modules of $\mathfrak{L}$. It is a nilpotent ideal, but not necessarily equal to the maximal nilpotent ideal. It is well known that $\text{Nrad}(\mathfrak{L}) = [\mathfrak{L}, \mathfrak{L}] \cap \mathfrak{r} = [\mathfrak{L}, \mathfrak{r}]$, and $\mathfrak{L}$ is reductive if and only if $\text{Nrad}(\mathfrak{L}) = 0$. Moreover, $\text{Nrad}(\mathfrak{L})$ is the minimal ideal in $\mathfrak{L}$ for which the quotient $\mathfrak{L}/\text{Nrad}(\mathfrak{L})$ is reductive. For proofs, see e.g. [Bou89, Chapter I, §5.3.].

Example 3. If $\mathfrak{L} = \mathfrak{g} \ltimes \mathfrak{r}$ is a reductive Lie algebra, then $\mathfrak{r}$ is precisely the center of $\mathfrak{L}$. If V is a simple $\mathfrak{g}$ -Harish-Chandra module for $\mathfrak{L}$, then, by Schur's lemma, $\mathfrak{r}$ acts by scalars on V . It follows that V is just a simple finite-dimensional $\mathfrak{L}$ -module. So, the notion of $\mathfrak{g}$ -Harish-Chandra modules is not very interesting for reductive Lie algebras.

Fix a positive part $\Delta^+(\mathfrak{g}, \mathfrak{h})$ in the root system $\Delta(\mathfrak{g}, \mathfrak{h})$, and a non-degenerate invariant symmetric bilinear form $\langle -, - \rangle$ on $\mathfrak{h}^*$. We have the classical triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Further, fix a weight $\delta \in \mathfrak{h}^*$ such that $\langle \delta, \alpha \rangle > 0$ for all $\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{h})$ and such that $\langle \delta, \alpha \rangle = 0$, for an integral weight α , implies $\alpha = 0$. Since $\mathfrak{L}$ is a finite-dimensional $\mathfrak{g}$ -module with respect to the adjoint action, it decomposes as a direct sum of its weight spaces $\mathfrak{L}_\mu$, where μ varies over the set of integral weights in $\mathfrak{h}^*$. Consider the following Lie subalgebras of $\mathfrak{L}$:

$$(1) \quad \tilde{\mathfrak{n}}_- := \bigoplus_{\langle \mu, \delta \rangle < 0} \mathfrak{L}_\mu, \quad \tilde{\mathfrak{h}} := \bigoplus_{\langle \mu, \delta \rangle = 0} \mathfrak{L}_\mu, \quad \tilde{\mathfrak{n}}_+ := \bigoplus_{\langle \mu, \delta \rangle > 0} \mathfrak{L}_\mu.$$

Note that this decomposition heavily depends on the choice of δ and not only on the choice of $\Delta^+(\mathfrak{g}, \mathfrak{h})$. However, for example, for truncated current Lie algebras (which include Takiff Lie algebras), the Schrödinger Lie algebra and conformal Galilei algebras, the decomposition (1) only depends on the choice of $\Delta^+(\mathfrak{g}, \mathfrak{h})$. From the construction, it is clear that $\tilde{\mathfrak{n}}_\pm \cap \mathfrak{g} = \mathfrak{n}_\pm$, and $\tilde{\mathfrak{h}} \cap \mathfrak{g} = \mathfrak{h}$. Moreover, from the condition prescribed on δ , it follows that $\tilde{\mathfrak{h}}$ is precisely the centralizer of $\mathfrak{h}$ in $\mathfrak{L}$. The decomposition $\mathfrak{L} = \tilde{\mathfrak{n}}_- \oplus \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}_+$ does not satisfy, in general, all the axioms in [Wil11, Section 2], since we do not require existence of any analogue of Chevalley involution (and even the dimensions of $\tilde{\mathfrak{n}}_-$ and $\tilde{\mathfrak{n}}_+$ might be different). However, it is good enough to define Verma modules with reasonable properties.

For an element $\lambda \in \tilde{\mathfrak{h}}^*$, denote the one-dimensional $\tilde{\mathfrak{b}} := \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}_+$ -module where $\tilde{\mathfrak{h}}$ acts as λ and $\tilde{\mathfrak{n}}_+$ acts trivially, by $\mathbb{C}_\lambda$. The *Verma module* with highest weight λ is defined

as

$$(2) \quad \Delta(\lambda) := \text{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_{\lambda} = U(\mathfrak{L}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\lambda} \cong U(\tilde{\mathfrak{n}}_-) \otimes_{\mathbb{C}} \mathbb{C}_{\lambda}.$$

Let $\Delta^{\pm}(\mathfrak{L}, \mathfrak{h})$ denote the set of all μ such that $\mathfrak{L}_{\mu} \neq 0$ and $\pm\langle\mu, \delta\rangle > 0$. We also set $\Gamma^{\pm} = \mathbb{Z}_{\geq 0} \Delta^{\pm}(\mathfrak{L}, \mathfrak{h})$. Recall that the *support* of a weight module is the set of all weights for which the corresponding weight spaces are non-zero. By the standard arguments using PBW theorem (cf. [Hum08]), we have:

Proposition 4. *The Verma module $\Delta(\lambda)$ is an $\mathfrak{h}$ -weight module, whose $\mathfrak{h}$ -support is $\lambda|_{\mathfrak{h}} + \Gamma^{-}$. The $\lambda|_{\mathfrak{h}}$ -weight space is one-dimensional, and $\Delta(\lambda)$ is generated by this weight vector, so any non-trivial quotient of $\Delta(\lambda)$ also has one-dimensional $\lambda|_{\mathfrak{h}}$ -weight space. Moreover, $\Delta(\lambda)$ has a unique simple quotient, which we denote by $\mathbf{L}(\lambda)$.*

For $\lambda \in \mathfrak{h}^*$, we denote by $\Delta^{\mathfrak{g}}(\lambda)$ the classical Verma module for $\mathfrak{g}$ with highest weight λ with respect to $\Delta^{+}(\mathfrak{g}, \mathfrak{h})$, and by $L(\lambda)$ the unique simple quotient of $\Delta^{\mathfrak{g}}(\lambda)$.

3. $\mathfrak{g}$ -HARISH-CHANDRA MODULES FOR TAKIFF LIE ALGEBRAS

3.1. Setup. Fix a finite-dimensional semi-simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. Define the associated *Takiff Lie algebra* $\mathfrak{T}$ as

$$\mathfrak{T} := \mathfrak{g} \otimes \mathbb{D},$$

where $\mathbb{D} = \mathbb{C}[x]/(x^2)$ is the algebra of dual numbers. The Lie bracket of $\mathfrak{T}$ is defined in the following way:

$$[v \otimes x^i, w \otimes x^j] := [v, w] \otimes x^{i+j}.$$

We identify $\mathfrak{g}$ with the subalgebra $\mathfrak{g} \otimes 1 \subseteq \mathfrak{T}$, and denote by $\bar{\mathfrak{g}} = \mathfrak{g} \otimes x \subseteq \mathfrak{T}$. Then $\bar{\mathfrak{g}}$ is a commutative ideal in $\mathfrak{T}$, and $\mathfrak{T} \cong \mathfrak{g} \ltimes \bar{\mathfrak{g}}$ (the semi-direct product given by the adjoint action of $\mathfrak{g}$ on $\bar{\mathfrak{g}}$). For $v \in \mathfrak{g}$, we denote by $\bar{v} = v \otimes x \in \bar{\mathfrak{g}}$.

Observe that the nilradical of $\mathfrak{T}$ is $\text{Nrad}(\mathfrak{T}) = [\mathfrak{T}, \bar{\mathfrak{g}}] = \bar{\mathfrak{g}}$. This means that $\bar{\mathfrak{g}}$ must necessarily annihilate any simple finite-dimensional $\mathfrak{T}$ -module.

In the triangular decomposition (1) for $\mathfrak{T}$, we have $\tilde{\mathfrak{h}} = \mathfrak{h} \oplus \bar{\mathfrak{h}}$ and $\tilde{\mathfrak{n}}_{\pm} = \mathfrak{n}_{\pm} \oplus \bar{\mathfrak{n}}_{\pm}$. We want to note that this is also a triangular decomposition in the sense of [Wil11]. A simplicity criterion for Verma modules over $\mathfrak{T}$ can be found in [Wil11, Theorem 7.1].

3.2. Purely Takiff part of the center. The universal enveloping algebra $U(\mathfrak{T})$ is free as a module over its center $Z(\mathfrak{T})$, see [Geo94, Geo95, FO05]. In case $\mathfrak{g}$ is of type A , algebraically independent generators of the center are given explicitly in [Mol].

Proposition 5. *There is an isomorphism of algebras*

$$(3) \quad Z(\mathfrak{g}) \cong Z(\mathfrak{T}) \cap U(\bar{\mathfrak{g}}).$$

Proof. This is clear since $U(\mathfrak{g}) \cong U(\bar{\mathfrak{g}})$ as $\mathfrak{g}$ -modules with respect to the adjoint action. By taking $\mathfrak{g}$ -invariants, we get (3). $\square$

It is easy to see that the isomorphism can be obtained by putting bars on all Lie algebra elements that appear in an expression in a fixed PBW-basis of elements from $Z(\mathfrak{g})$. Hence we denote the right-hand side of (3) by $\overline{Z(\mathfrak{g})}$. This will be referred to as the *purely Takiff part of the center* $Z(\mathfrak{T})$. The full center $Z(\mathfrak{T})$ is in general bigger than $\overline{Z(\mathfrak{g})}$, see [Mol].

3.3. Universal modules. Assume that the simple $\mathfrak{g}$ -module $L(\lambda)$ is finite-dimensional, that is $\lambda \in \mathfrak{h}^*$ is integral, dominant and regular. Define

$$Q(\lambda) := \text{Ind}_{\mathfrak{g}}^{\mathfrak{T}} L(\lambda) = U(\mathfrak{T}) \otimes_{U(\mathfrak{g})} L(\lambda) \cong U(\bar{\mathfrak{g}}) \otimes_{\mathbb{C}} L(\lambda).$$

Note that we have (see e.g. [Kna88, Proposition 6.5.])

$$(4) \quad Q(\lambda) \cong Q(0) \otimes L(\lambda),$$

where we consider $L(\lambda)$ as a $\mathfrak{T}$ -module with the trivial $\bar{\mathfrak{g}}$ -action.

Proposition 6. *We have the following isomorphism of algebras:*

$$\text{End}(Q(0)) \cong \overline{Z(\mathfrak{g})}.$$

Proof. The module $Q(0)$ is generated by $1 \otimes 1$ by construction, so any endomorphism of $Q(0)$ is uniquely determined by the image of $1 \otimes 1$. Denote this image by $u \otimes 1$, for some $u \in U(\bar{\mathfrak{g}})$. The element $u \otimes 1$ generates the trivial $\mathfrak{g}$ -submodule (since $1 \otimes 1$ does), so u must commute with $\mathfrak{g}$. Of course, u commutes with $\bar{\mathfrak{g}}$. Hence $u \in Z(\mathfrak{T}) \cap U(\bar{\mathfrak{g}}) = \overline{Z(\mathfrak{g})}$.

Conversely, any $u \in \overline{Z(\mathfrak{g})}$, being central, defines an endomorphism of $Q(0)$. This endomorphism maps $1 \otimes 1$ to $u \otimes 1$. The claim follows. $\square$

For an algebra homomorphism $\chi: \overline{Z(\mathfrak{g})} \rightarrow \mathbb{C}$, consider the corresponding *universal module*

$$Q(\lambda, \chi) := Q(\lambda) / \mathfrak{m}_{\chi} Q(\lambda),$$

where $\mathfrak{m}_{\chi}$ is the maximal ideal in $\overline{Z(\mathfrak{g})}$ corresponding to χ . On $Q(\lambda, \chi)$, the purely Takiff part of the center acts via the scalars prescribed by χ . Observe that from (4) and the right-exactness of tensor product we have

$$(5) \quad Q(\lambda, \chi) \cong Q(0, \chi) \otimes L(\lambda).$$

For finite-dimensional simple $\mathfrak{g}$ -modules $L(\mu)$, $L(\nu)$ and $L(\lambda)$, denote by $l_{\nu, \lambda}^{\mu}$ the Littlewood-Richardson coefficient, i.e., the multiplicity of $L(\mu)$ in $L(\nu) \otimes L(\lambda)$.

Proposition 7. (a) *Let λ, χ be as before. The module $Q(\lambda, \chi)$ is a $\mathfrak{g}$ -Harish-Chandra module, and the multiplicities are given as follows:*

$$(6) \quad [Q(\lambda, \chi): L(\mu)] = \sum_{\nu} \dim L(\nu)_0 \cdot l_{\nu, \lambda}^{\mu} < \infty.$$

(b) *Let V be any simple $\mathfrak{T}$ -module that has some finite-dimensional $L(\lambda)$ as a simple $\mathfrak{g}$ -submodule. Then V is a quotient of $Q(\lambda, \chi)$ for a unique χ . In particular, V is a $\mathfrak{g}$ -Harish-Chandra module, and (6) gives an upper bound for the multiplicities of its $\mathfrak{g}$ -types.*

Proof. (a) Suppose first that $\lambda = 0$. Then, as a $\mathfrak{g}$ -module, $Q(0)$ is isomorphic to $U(\mathfrak{g})$ with respect to the adjoint action. Taking the χ -component of $Q(0)$ corresponds to factoring $U(\mathfrak{g})$ by the ideal generated by the corresponding central character of $Z(\mathfrak{g})$. From Kostant's theorem (see [Jan83, Section 3.1]), it follows that $Q(0)$ decomposes as direct sum of finite-dimensional $\mathfrak{g}$ -submodules, and that $[Q(0, \chi): L(\mu)] = \dim L(\mu)_0$. The general statement now follows from (5).

Note that the value in (6) is finite, since, for fixed μ and λ , the value $l_{\nu, \lambda}^{\mu}$ is non-zero only for finitely many ν .

(b) This follows from Schur's Lemma by adjunction. $\square$

Corollary 8. *Given χ , there exists a unique simple $\mathfrak{T}$ -module V which contains $L(0)$ as a $\mathfrak{g}$ -submodule and has the Takiff part of the central character equal to χ . Moreover, V is a $\mathfrak{g}$ -Harish-Chandra module.*

Furthermore, if χ does not correspond to the trivial $\mathfrak{T}$ -module, then V is infinite-dimensional.

Proof. By Proposition 7, the module $Q(0, \chi)$ has a unique occurrence of $L(0)$, and is generated by it. Therefore, the sum of all its submodules not containing $L(0)$ as a composition factor is the unique maximal submodule; denote it by N . It follows that $V := Q(0, \chi)/N$ is the unique simple quotient of $Q(0, \chi)$.

Suppose now that V is finite-dimensional. The nilradical $\bar{\mathfrak{g}}$ must act trivially on it. Because of simplicity, we must have $V = L(0)$, which is a contradiction. $\square$

Conjecture 9. *For a “generic” χ , the module $Q(0, \chi)$ is simple.*

We will prove this conjecture for the Takiff $\mathfrak{sl}_2$ case in Section 4. We will also prove it for the Schrödinger Lie algebra in Section 5 (but, strictly speaking, it is not a special instance of the above conjecture). This is the starting point in our classification of $\mathfrak{g}$ -Harish-Chandra modules for these Lie algebras.

4. $\mathfrak{sl}_2$ -HARISH-CHANDRA MODULES FOR THE TAKIFF $\mathfrak{sl}_2$

4.1. Setup. For this section, we fix the Takiff Lie algebra associated to $\mathfrak{g} := \mathfrak{sl}_2$:

$$\mathfrak{T} = \mathfrak{sl}_2 \otimes \mathbb{D} = \mathfrak{sl}_2 \ltimes \overline{\mathfrak{sl}_2}.$$

We use the usual notation f, h, e for the standard basis elements of $\mathfrak{sl}_2$, and $\bar{f}, \bar{h}, \bar{e}$ for their counterparts in the ideal $\overline{\mathfrak{sl}_2}$.

Our classification of simple $\mathfrak{g}$ -Harish-Chandra modules for the Takiff $\mathfrak{sl}_2$ should be, of course, deducible from the classification of all simple weight modules given in [BL17]. However, our approach is completely different and, unlike the approach of [BL17], has clear potential for generalization to other Lie algebras. Also, our description of simple $\mathfrak{g}$ -Harish-Chandra modules is much more explicit, and it provides a connection to highest weight theory for $\mathfrak{T}$ and utilizes the use of analogues of projective functors for $\mathfrak{T}$.

The center $Z(\mathfrak{T})$ is a polynomial algebra generated by two algebraically independent elements (see [Mol]):

$$(7) \quad \begin{aligned} C &= \bar{h}h + 2\bar{f}e + 2\bar{e}f, \\ \bar{C} &= \bar{h}^2 + 4\bar{f}\bar{e}. \end{aligned}$$

The purely Takiff part of the center is, of course, $\overline{Z(\mathfrak{g})} = \mathbb{C}[\bar{C}]$. So, a homomorphism $\chi: \overline{Z(\mathfrak{g})} \rightarrow \mathbb{C}$ is uniquely determined by the value $\chi(\bar{C})$, which can be an arbitrary complex number. In the remainder, we write χ for $\chi(\bar{C})$, for the sake of brevity.

4.2. Universal modules. We can describe $Q(0, \chi)$ very explicitly.

Lemma 10. (a) *As $\mathfrak{T}$ -modules, we have $Q(0) \cong U(\bar{\mathfrak{g}})$ and $Q(0, \chi) \cong U(\bar{\mathfrak{g}})/(\bar{C} - \chi)$, where $\mathfrak{g}$ acts by the adjoint action, and $\bar{\mathfrak{g}}$ by the left multiplication.*

(b) *The set $\{\bar{f}^i \bar{h}^\epsilon \bar{e}^j : i, j \geq 0, \epsilon \in \{0, 1\}\}$ is a basis for $Q(0, \chi)$.*

(c) *As a $\mathfrak{g}$ -module, $Q(0, \chi) \cong \bigoplus_{k \geq 0} L(2k)$. Moreover, $\bar{e}^k$ is the highest weight vector in $L(2k)$.*

(d) C acts as zero on $Q(0)$ and on every $Q(0, \chi)$.

Proof. The first claim is clear. The second one follows from the PBW basis in $U(\mathfrak{g})$ and the relation $\bar{h}^2 = -4\bar{f}\bar{e} + \chi$ in the quotient.

The decomposition in the third claim is given by Kostant's theorem, see [Jan83, Section 3.1]. Since $\bar{e}^k$ is of weight $2k$ and annihilated by e , it must be highest weight vector of a $\mathfrak{g}$ -submodule isomorphic to $L(2k)$, which, we know, occurs uniquely in $Q(0, \chi)$.

The last claim follows from the definitions by a direct calculation. $\square$

The action of $\mathfrak{T}$ on $U(\bar{\mathfrak{g}})$ and its quotients will be denoted by $\circ$, in order not to confuse it with the multiplication $\cdot$ in the enveloping algebra. These coincide for $\bar{\mathfrak{g}}$ but not for $\mathfrak{g}$, where the action is adjoint. Note that $U(\bar{\mathfrak{g}})$ is not closed under the left multiplication with the whole $\mathfrak{T}$.

Theorem 11. *The module $Q(0, \chi)$ is simple if and only if $\chi \neq 0$.*

The module $Q(0, 0)$ has infinite length, and a $\mathfrak{T}$ -filtration whose composition factors are $L(0), L(2), L(4) \dots$ with the trivial action of $\bar{\mathfrak{g}}$.

Proof. Assume $\chi \neq 0$, and let $V \subseteq Q(0, \chi)$ be non-zero submodule. Take k to be the smallest non-negative integer such that $L(2k) \subseteq V$. If $k = 0$, then $V = Q(0, \chi)$ since $L(0)$ generates $Q(0, \chi)$, and we are done. So, let us assume now $k \geq 1$. We have $\bar{e}^k \in V$, so if we find an element from $U(\mathfrak{T})$ that maps $\bar{e}^k \in V$ to $\bar{e}^{k-1}$, we will get a contradiction. That element can be taken as $\frac{1}{k\chi}(4k\bar{f} - \bar{h}f)$, namely:

$$\begin{aligned} (4k\bar{f} - \bar{h}f) \circ \bar{e}^k &= 4k\bar{f}\bar{e}^k - \bar{h}[f, \bar{e}^k] \\ &= 4k\bar{f}\bar{e}^k + k\bar{h}^2\bar{e}^{k-1} \\ &= 4k\bar{f}\bar{e}^k + k(-4\bar{f}\bar{e} + \chi)\bar{e}^{k-1} \\ &= k\chi\bar{e}^{k-1}. \end{aligned}$$

We conclude that $Q(0, \chi)$ is simple.

For the converse, assume $\chi = 0$. We will show that for any $k \geq 0$, the subspace $Q_k := \bigoplus_{t \geq k} L(2t)$ is a submodule. From this, the theorem will follow.

Let us first prove that $L(2k)$ is equal to the span of $\{\bar{f}^i \bar{h}^\epsilon \bar{e}^j : \epsilon \in \{0, 1\}, i + \epsilon + j = k\}$. This set contains $\bar{e}^k$, so it is enough to see that it is stable under f . We calculate the two cases whether ϵ is 0 or 1 separately:

$$\begin{aligned} f \circ \bar{f}^i \bar{e}^j &= \bar{f}^i [f, \bar{e}^j] = -j\bar{f}^i \bar{h} \bar{e}^{j-1}, \\ f \circ \bar{f}^i \bar{h} \bar{e}^j &= \bar{f}^i [f, \bar{h}] \bar{e}^j + \bar{f}^i \bar{h} [f, \bar{e}^j] \\ &= 2\bar{f}^{i+1} \bar{e}^j - j\bar{f}^i \bar{h}^2 \bar{e}^{j-1} \\ &= 2\bar{f}^{i+1} \bar{e}^j + 4j\bar{f}^{i+1} \bar{e}^j \\ &= (4j + 2)\bar{f}^{i+1} \bar{e}^j. \end{aligned}$$

From this description of $L(2k)$, one easily checks that $\bar{f}, \bar{h}, \bar{e}$ map $L(2k)$ to $L(2k+2)$. From this, it follows that Q_k is a submodule. $\square$

Remark 12 (A sketch of an alternative proof of simplicity of $Q(0, \chi)$ for $\chi \neq 0$). Suppose V is a $\mathfrak{T}$ -submodule of $Q(0, \chi)$ containing $L(2k)$, with $k > 0$ minimal. By

applying $\bar{e}$, we see that, as a $\mathfrak{g}$ -module, $V \cong L(2k) \oplus L(2k+2) \oplus \dots$. This implies that the quotient

$$Q(0, \chi)/V \cong L(0) \oplus L(2) \oplus \dots \oplus L(2k-2)$$

is simple as a $\mathfrak{T}$ -module and is finite-dimensional. Since $\bar{C}$ consists of elements from the nilradical of $\mathfrak{T}$, it must act as zero on this quotient. This is a contradiction with $\chi \neq 0$.

To classify simple $\mathfrak{g}$ -Harish-Chandra modules, by (5) and Proposition 7(b) we should find all simple quotients of all tensor products of $Q(0, \chi)$ with finite-dimensional $\mathfrak{g}$ -modules. It is not easy to do this directly, so we establish a connection with Verma modules, and perform calculations there.

4.3. Verma modules. Verma modules for the Takiff $\mathfrak{sl}_2$ are studied in detail in [MS19]. Recall (2) and Proposition 4. Also recall that $\tilde{\mathfrak{h}} = \mathfrak{h} \oplus \bar{\mathfrak{h}}$ and $\tilde{\mathfrak{n}}_{\pm} = \mathfrak{n}_{\pm} \oplus \bar{\mathfrak{n}}_{\pm}$. For a weight $\lambda \in \tilde{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \bar{\mathfrak{h}}^*$, we denote $\lambda_1 := \lambda(h)$ and $\lambda_2 := \lambda(\bar{h})$.

Proposition 13 ([MS19, Proposition 1] or [Wil11, Theorem 7.1]). *The Verma module $\Delta(\lambda)$ is simple if and only if $\lambda_2 \neq 0$.*

The generators of the center C and $\bar{C}$ act on the Verma module $\Delta(\lambda)$ as the scalars $\lambda_2(\lambda_1 + 2)$ and λ_2^2 respectively, see (7). Therefore, with our convection, $\chi = \lambda_2^2$.

Lemma 14. *Non-isomorphic Verma modules $\Delta(\lambda)$ and $\Delta(\lambda')$ have the same central character if and only if either $\lambda'_2 = \lambda_2 = 0$, or $\lambda'_2 = -\lambda_2 \neq 0$ and $\lambda'_1 = -\lambda_1 - 4$.*

Proof. From the explicit description of generators of the center, we get a system of equations

$$\begin{cases} \lambda_2(\lambda_1 + 2) = \lambda'_2(\lambda'_1 + 2) \\ \lambda_2^2 = (\lambda'_2)^2 \end{cases}.$$

which is easily solved. □

Denote by $\Delta^{\mathfrak{g}}(\mu) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\mu}$ the classical Verma module for $\mathfrak{g}$ with highest weight $\mu \in \mathbb{C}$, and by $P^{\mathfrak{g}}(\mu)$ its indecomposable projective cover in the category $\mathcal{O}$ for $\mathfrak{g}$. Recall that, if $\mu \in \mathbb{Z}_{\geq 0}$, $P^{\mathfrak{g}}(-\mu - 2)$ is the unique non-trivial extension of $\Delta^{\mathfrak{g}}(-\mu - 2)$ by $\Delta^{\mathfrak{g}}(\mu)$, and that there are no extensions between other $\Delta^{\mathfrak{g}}$'s (inside category $\mathcal{O}$).

Lemma 15. *As a $\mathfrak{g}$ -module, $\Delta(\lambda)$ has a filtration with subquotients isomorphic to $\Delta^{\mathfrak{g}}(\lambda_1 - 2k)$, $k = 0, 1, 2, \dots$*

If $\lambda_2 = 0$ or $\lambda_1 \notin \mathbb{Z}_{\geq 0}$, then, as a $\mathfrak{g}$ -module, we have

$$\Delta(\lambda) \cong \bigoplus_{k \geq 0} \Delta^{\mathfrak{g}}(\lambda_1 - 2k).$$

Otherwise (i.e. if $\lambda_2 \neq 0$ and $\lambda_1 \in \mathbb{Z}_{\geq 0}$), we have, as $\mathfrak{g}$ -modules,

$$\Delta(\lambda) \cong \begin{cases} \bigoplus_{k=1}^{\frac{\lambda_1}{2}+1} P^{\mathfrak{g}}(-2k) \oplus \bigoplus_{k \geq 2} \Delta^{\mathfrak{g}}(-\lambda_1 - 2k) & : \lambda_1 \text{ even,} \\ \Delta^{\mathfrak{g}}(-1) \oplus \bigoplus_{k=1}^{\frac{\lambda_1+1}{2}} P^{\mathfrak{g}}(-2k-1) \oplus \bigoplus_{k \geq 2} \Delta^{\mathfrak{g}}(-\lambda_1 - 2k) & : \lambda_1 \text{ odd.} \end{cases}$$

Proof. Denote by v_λ a basis element of $\mathbb{C}_\lambda$. Then $\Delta(\lambda)$ has a basis of weight vectors $\{f^i \bar{f}^j v_\lambda : i, j \geq 0\}$. A direct computation (with help of [Hum72, Lemma 21.2] and its Takiff analogue, alternatively use [CSZ16, Lemma 2.1]) shows that

$$(8) \quad \begin{aligned} e \cdot f^i \bar{f}^j v_\lambda &= [e, f^i] \bar{f}^j v_\lambda + f^i [e, \bar{f}^j] v_\lambda \\ &= i(\lambda_1 - i - 2j + 1) f^{i-1} \bar{f}^j v_\lambda + j \lambda_2 f^i \bar{f}^{j-1} v_\lambda. \end{aligned}$$

This implies that the required filtration is given by the degree of $\bar{f}$. The subquotients are given by the span of $\{f^i \bar{f}^k v_\lambda : i \geq 0\}$, which is clearly isomorphic to $\Delta^{\mathfrak{g}}(\lambda_1 - 2k)$.

If $\lambda_2 = 0$, it is clear that the span of $\{f^i \bar{f}^k v_\lambda : i \geq 0\}$, k fixed, is a $\mathfrak{g}$ -submodule. If $\lambda_1 \notin \mathbb{Z}_{\geq 0}$, then there are no possible non-trivial extensions between $\Delta^{\mathfrak{g}}(\lambda_1 - 2k)$, $k \geq 0$, hence $\Delta(\lambda)$ splits into as a direct sum of these.

Suppose now $\lambda_2 \neq 0$ and $\lambda_1 \in \mathbb{Z}_{\geq 0}$. Fix $\mu \in \{0, 1, \dots, \lambda_1\}$ of the same parity as λ_1 . It is enough to show that $\Delta^{\mathfrak{g}}(-\mu - 2)$ is not a $\mathfrak{g}$ -submodule of $\Delta(\lambda)$. Suppose it is. Its highest weight vector $v_{-\mu-2}$ must be a non-trivial linear combination of $f^i \bar{f}^j v_\lambda$ with $i + j = \frac{\lambda_1 + \mu}{2} + 1 =: t$, with a non-zero coefficient by $\bar{f}^t v_\lambda$.

From (8) it follows that the matrix of e in bases $f^i \bar{f}^{t-i} v_\lambda$, $i = 0, \dots, t$, and $f^i \bar{f}^{t-1-i} v_\lambda$, $i = 0, \dots, t-1$, has the form

$$\begin{pmatrix} * & * & 0 & \dots & 0 & 0 & 0 \\ 0 & * & * & \dots & 0 & 0 & 0 \\ 0 & 0 & * & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & * & 0 & 0 \\ 0 & 0 & 0 & \dots & * & * & 0 \\ 0 & 0 & 0 & \dots & 0 & * & * \end{pmatrix},$$

with all $*$ non-zero, except the one on the position $(\mu + 1, \mu + 2)$, where we have zero (because the bracket in (8) is zero for $f^{\mu+1} \bar{f}^{\frac{\lambda_1 - \mu}{2}} v_\lambda$). From this, it follows that e cannot annihilate $v_{-\mu-2}$, a contradiction. $\square$

Lemma 16. For $\lambda_2 \neq 0$ and $\mu \in \mathbb{Z}_{\geq 0}$ there is an isomorphism of $\mathfrak{T}$ -modules

$$\Delta(\lambda) \otimes L(\mu) \cong \Delta(\lambda_1 + \mu, \lambda_2) \oplus \Delta(\lambda_1 + \mu - 2, \lambda_2) \oplus \dots \oplus \Delta(\lambda_1 - \mu, \lambda_2).$$

Proof. In the same way as for the semi-simple case, see e.g. [Hum08, 6.3.] one sees that the left-hand side has a filtration with subquotients equal to the summands on the right-hand side. But these subquotients have different central characters, which follows from Lemma 14, so they split. $\square$

4.4. Enright-Arkhipov completion. Here we show that $\mathfrak{g}$ -Harish-Chandra modules naturally occur in a certain completion (or localization) of Verma modules. We consider a combination of two of such constructions, originally given by Enright in [Enr79], and Arkhipov in [Ark04]. See also [Deo80, AS03, KM02, KM05]. To ease the notation a little bit, we will write U instead of $U(\mathfrak{T})$ for the rest of this section.

Fix an ad-nilpotent element $x \in \mathfrak{T}$ (for example f , e , or $\bar{e}$, which we will use), and denote by $U_{(x)}$ the localization of the algebra U by the multiplicative set generated by x . This localization satisfies the Ore conditions by [Mat00, Lemma 4.2.], but this is also visible from the proof of Lemma 32. Since U has no zero-divisors, the canonical map $U \rightarrow U_{(x)}$ is injective. Hence we may consider the U - $U_{(x)}$ -bimodule

$$S_x := U_{(x)} / U.$$

Lemma 17. (a) Suppose $\{x, x_1, \dots, x_5\}$ is a basis for $\mathfrak{T}$. The set of all monomials $\{x^k x_1^{k_1} \dots x_5^{k_5} : k \in \mathbb{Z}, k_1, \dots, k_5 \in \mathbb{Z}_{\geq 0}\}$ is a basis for $U_{(x)}$.

(b) The analogous set, but with $k \in \mathbb{Z}_{<0}$, is a basis for the quotient S_x .

Proof. The set in the first claim is a generating set for $U_{(x)}$, which follows from PBW and the properties of Ore localization. But this set is also linearly independent, since for its any finite subset, the multiplication from the left by x^m for some large m produces a linearly independent set in $U \leq U_{(x)}$. This proves the first claim and the second claim follows from it. $\square$

Denote by $j: M \rightarrow U_{(x)} \otimes_U M$ the canonical map. By using the right exactness of the tensor product, we can identify

$$(9) \quad S_x \otimes_U M \cong \left(U_{(x)} \otimes_U M \right) / j(M).$$

Moreover, if M is a $\mathfrak{T}$ -module on which x acts injectively, then the canonical map j is injective. In particular, this is true if M is a Verma module $\Delta(\lambda)$ and $x = f$.

Lemma 18 ([Deo80, AS03]). Fix $x \in \{f, e, \bar{f}, \bar{e}\}$, let M be $\mathfrak{T}$ -module, and L a finite-dimensional $\mathfrak{T}$ -module. Then there is a natural isomorphism of $\mathfrak{T}$ -modules

$$S_x \otimes_U (M \otimes L) \cong \left(S_x \otimes_U M \right) \otimes L.$$

Proof. There is an isomorphism $U_{(x)} \otimes_U (M \otimes L) \rightarrow (U_{(x)} \otimes_U M) \otimes L$ given by

$$x^{-n} \otimes (m \otimes v) \mapsto \sum_{k \geq 0} (-1)^k \binom{n+k-1}{k} (x^{-n-k} \otimes m) \otimes x^k v,$$

with the inverse given by $(x^{-n} \otimes m) \otimes v \mapsto x^{-ar} \otimes \sum_{k \geq 0} \binom{ar}{k} (x^{ar-n-k} m \otimes x^k v)$, where $r, a \in \mathbb{Z}_{>0}$ are chosen so that x^r annihilates L and $(r-1)a \geq n$. This is proved in [Deo80, Theorem 3.1] and [AS03, Theorem 3.2.] for the semi-simple case, but the proof is analogous in general. In proving that these maps compose to the identity, the following combinatorial formula is helpful: $\sum_{k=0}^n (-1)^k \binom{a}{n-k} \binom{b+k}{k} = \binom{a-b-1}{n}$.

One can check that these isomorphisms preserve the canonical images of $M \otimes L$ in both sides, see (9), so they induce the required isomorphisms on the quotients. $\square$

For a $\mathfrak{T}$ -module M , we write ${}^x M$ for the set of all elements $m \in M$ for which the action of x is locally finite, in the sense that $\dim \mathbb{C}[x]m < \infty$. Note that this is a variant of the Zuckerman functor.

Lemma 19. For a $\mathfrak{T}$ -module M , ${}^x M$ is a $\mathfrak{T}$ -submodule. Moreover, the assignment $M \mapsto {}^x M$ is a left-exact functor in the obvious way.

Proof. Since x is assumed to be ad-nilpotent, the claim follows from the formula in [Hum72, Lemma 21.4.]. $\square$

Definition 20. For a $\mathfrak{T}$ -module M , define

$$(10) \quad \mathbf{EA}(M) := \left(S_f \otimes_U M \right)^e.$$

This is a functor on the category of $\mathfrak{T}$ -modules in the obvious way, which we call Enright-Arkhipov's completion functor.

Proposition 21. *The functor $\mathbf{EA}$ commutes with tensoring with a finite-dimensional $\mathfrak{T}$ -module. More precisely, let M be a $\mathfrak{T}$ -module, and L a finite-dimensional $\mathfrak{T}$ -module. Then there is a natural isomorphism of $\mathfrak{T}$ -modules*

$$\mathbf{EA}(M \otimes L) \cong \mathbf{EA}(M) \otimes L.$$

Proof. Because of Lemma 18, it is enough to show that ${}^e(M \otimes L) = ({}^eM) \otimes L$ for $\mathfrak{g}$ -modules M and L with $L = L(\mu)$ simple finite-dimensional. This is proved in [Deo80, Corollary 3.2], but we also give a proof for the sake of completeness.

The inclusion $({}^eM) \otimes L \subseteq {}^e(M \otimes L)$ is trivial. For the converse, denote by v the lowest weight vector of L . Then $v, ev, \dots, e^\mu v$ is a basis for L . Take a general element $m = \sum_{i=0}^\mu m_i \otimes e^i v \in {}^e(M \otimes L)$, and observe that for $n > \mu$ we have

$$\begin{aligned} e^n \cdot m &= \sum_{i=0}^\mu \sum_{j=0}^{\mu-i} \binom{n}{j} e^{n-j} m_i \otimes e^{i+j} v \\ &= \sum_{i=0}^\mu \left(e^n m_i + \sum_{j=0}^{i-1} \binom{n}{i-j} e^{n+j-i} m_j \right) \otimes e^i v. \end{aligned}$$

For a fixed i , the vectors inside the big brackets must span a finite-dimensional space when n varies. From this, by an induction on i follows that $e^n m_i$ span a finite-dimensional space, hence $m \in ({}^eM) \otimes L$. $\square$

Example 22. Let us consider $\Delta^{\mathfrak{g}}(\mu)$, with $\mu \in \mathbb{C}$. From Lemma 17 it follows that the set $\{f^{-k} v_\mu : k > 0\}$ is a basis for $S_f \otimes_U M$ (and from an argument for linear independence very similar to the one in the proof of Lemma 17). One can easily prove by induction the following commutation relations (similar to [Maz10, 3.5]):

$$\begin{aligned} (11) \quad [h, f^{-k}] &= 2k f^{-k}, \\ [e, f^{-k}] &= -k f^{-k-1} (h + k + 1). \end{aligned}$$

From this, it is not hard to see that

$$\mathbf{EA}(\Delta^{\mathfrak{g}}(\mu)) \cong \begin{cases} L(-\mu - 2) & : \mu \in \mathbb{Z} \text{ and } \mu \leq -2, \\ 0 & : \text{otherwise.} \end{cases}$$

Similarly, one sees that $\mathbf{EA}(P^{\mathfrak{g}}(\mu)) = 0$ for $\mu \in \mathbb{Z}$ and $\mu \leq -2$. (Or using the fact that big projective modules can be obtained by tensoring dominant Verma modules with finite-dimensional modules, together with Proposition 21).

Recall that we use notation $\lambda = (\lambda_1, \lambda_2) \in \tilde{\mathfrak{h}}^*$, with $\lambda_1 = \lambda(h)$ and $\lambda_2 = \lambda(\bar{h})$.

Theorem 23. *Take λ with $\lambda_1 \in \mathbb{Z}$ and $\lambda_2 \neq 0$. Then $\mathbf{EA}(\Delta(\lambda))$ is a simple $\mathfrak{g}$ -Harish-Chandra module. As a $\mathfrak{g}$ -module, it decomposes as follows:*

$$(12) \quad \mathbf{EA}(\Delta(\lambda)) \cong \bigoplus_{k \geq 0} L(|\lambda_1 + 2| + 2k).$$

Proof. Lemma 15, the fact that the functor $\mathbf{EA}$ commutes with the forgetful functor from $\mathfrak{T}$ -modules to $\mathfrak{g}$ -modules, and Example 22 imply (12).

From Lemma 17 we have a basis for $S_f \otimes_U \Delta(\lambda)$ consisting of $f^{-i} \bar{f}^j v_\lambda$, for $i \geq 1$ and $j \geq 0$. Since the lowest weight vector of a $\mathfrak{g}$ -type $L(\mu)$ (μ of the same parity as λ_1) inside $\mathbf{EA}(\Delta(\lambda))$ must be annihilated by f , it must be (up to scalar) equal to $f^{-1} \bar{f}^t v_\lambda$, where $t = \frac{\mu + \lambda_1}{2} + 1$.

Now we will prove that $\mathbf{EA}(\Delta(\lambda))$ is simple. Let V be its non-zero submodule, and suppose it contains $L(\mu)$ for some μ from (12). By applying $\bar{f}$ on $f^{-1}\bar{f}^t v_\lambda$, we get that $L(\mu + 2k) \subseteq V$ for all $k \geq 0$.

To prove that $V = \mathbf{EA}(\Delta(\lambda))$, it is enough to assume $\mu > |\lambda_1 + 2|$ and to find an element in $U(\mathfrak{T})$ that maps $f^{-1}\bar{f}^t v_\lambda$ to $f^{-1}\bar{f}^{t-1} v_\lambda$.

In addition to (11), we will use the following commutation relations, whose proofs are analogous to the ones for (11):

$$(13) \quad \begin{aligned} [\bar{h}, f^{-k}] &= 2kf^{-k-1}\bar{f}, \\ [\bar{e}, f^{-k}] &= -kf^{-k-1}\bar{h} - k(k+1)f^{-k-2}\bar{f}. \end{aligned}$$

From this, we have:

$$\begin{aligned} e \cdot f^{-1}\bar{f}^t v_\lambda &= [e, f^{-1}]\bar{f}^t v_\lambda + f^{-1}[e, \bar{f}^t]v_\lambda \\ &= -f^{-2}(h+2)\bar{f}^t v_\lambda + t\lambda_2 f^{-1}\bar{f}^{t-1} v_\lambda \\ &= \mu f^{-2}\bar{f}^t v_\lambda + t\lambda_2 f^{-1}\bar{f}^{t-1} v_\lambda, \end{aligned}$$

$$\begin{aligned} \bar{h}e \cdot f^{-1}\bar{f}^t v_\lambda &= \mu\bar{h}f^{-2}\bar{f}^t v_\lambda + t\lambda_2\bar{h}f^{-1}\bar{f}^{t-1} v_\lambda \\ &= \mu[\bar{h}, f^{-2}]\bar{f}^t v_\lambda + \mu\lambda_2 f^{-2}\bar{f}^t v_\lambda + t\lambda_2[\bar{h}, f^{-1}]\bar{f}^{t-1} v_\lambda + t\lambda_2^2 f^{-1}\bar{f}^{t-1} v_\lambda \\ &= 4\mu f^{-3}\bar{f}^{t+1} v_\lambda + (\lambda_1 + 2\mu + 2)\lambda_2 f^{-2}\bar{f}^t v_\lambda + t\lambda_2^2 f^{-1}\bar{f}^{t-1} v_\lambda, \end{aligned}$$

$$\begin{aligned} 2\mu\bar{e} \cdot f^{-1}\bar{f}^t v_\lambda &= 2\mu[\bar{e}, f^{-1}]\bar{f}^t v_\lambda \\ &= -4\mu f^{-3}\bar{f}^{t+1} v_\lambda - 2\mu\lambda_2 f^{-2}\bar{f}^t v_\lambda. \end{aligned}$$

From this it follows that

$$(\bar{h}e - 2\mu\bar{e}) \cdot f^{-1}\bar{f}^t v_\lambda = (\lambda_1 + 2)\lambda_2 f^{-2}\bar{f}^t v_\lambda + t\lambda_2^2 f^{-1}\bar{f}^{t-1} v_\lambda.$$

Now we claim that a non-trivial linear combination of e and $(\bar{h}e - 2\mu\bar{e})$ will map $f^{-1}\bar{f}^t v_\lambda$ to $f^{-1}\bar{f}^{t-1} v_\lambda$. This is true, because the determinant

$$\begin{vmatrix} \mu & (\lambda_1 + 2)\lambda_2 \\ t\lambda_2 & t\lambda_2^2 \end{vmatrix} = \lambda_2^2(\mu + \lambda_1 + 2)(\mu - \lambda_1 - 2) \neq 0.$$

This finishes the proof of simplicity. $\square$

Remark 24 (A sketch of an alternative proof of simplicity of $\mathbf{EA}(\Delta(\lambda))$ for $\lambda_1 \in \mathbb{Z}$ and $\lambda_2 \neq 0$). Suppose V is a submodule of $\mathbf{EA}(\Delta(\lambda))$ having $L(\mu)$, $\mu > |\lambda_1 + 2|$ minimal. By applying $\bar{f}$, we see that as a $\mathfrak{g}$ -module, $V \cong L(\mu) \oplus L(\mu + 2) \oplus \dots$. This implies that the quotient $\mathbf{EA}(\Delta(\lambda))/V \cong L(|\lambda_1 + 2|) \oplus L(|\lambda_1 + 2| + 2) \oplus \dots \oplus L(\mu - 2)$ is simple as a $\mathfrak{T}$ -module and finite-dimensional. Since $\bar{C}$ consists of elements from the nilradical of $\mathfrak{T}$, it must act as zero on this quotient. But $\bar{C}$ is central, so it still acts as λ_2^2 on the localization, a contradiction.

4.5. Classification. In this subsection, we use the relation with highest weight theory established above to classify all simple $\mathfrak{g}$ -Harish-Chandra modules for $\mathfrak{T}$. It will be more convenient to shift the notation for the first parameter in our modules by -2 .

Definition 25. For $n \in \mathbb{Z}$ and $\lambda_2 \neq 0$, denote

$$V(n, \lambda_2) := \mathbf{EA}(\Delta(n - 2, \lambda_2)).$$

Corollary 26. The module $V(n, \lambda_2)$ is a simple $\mathfrak{g}$ -Harish-Chandra module, it has $\mathfrak{g}$ -types $L(|n|), L(|n| + 2), L(|n| + 4) \dots$ and each of these occurs with multiplicity one.

If $V(n, \lambda_2) \cong V(n', \lambda'_2)$, then $(n', \lambda'_2) = (n, \lambda_2)$ or $(-n, -\lambda_2)$.

Proof. The first statement follows from Theorem 23.

It is clear that the functor $\mathbf{EA}$ preserves central character. So, the generators of the center C and $\bar{C}$ act as the scalars $n\lambda_2$ and λ_2^2 , respectively. From this the second statement follows. $\square$

We will see later in this subsection that the modules $V(n, \lambda_2)$ exhaust all infinite-dimensional simple $\mathfrak{g}$ -Harish-Chandra modules.

On Figure 1 to 4 we present several $V(n, \lambda_2)$'s, and how they are constructed. The gray area on the left hand-side is the Verma module, decomposed into rows according to Lemma 15, and furthermore, into weight spaces. The remaining bullets represent S_f tensored with the Verma module. The arrows represent non-zero action of e , and the light-gray area on the right hand-side contains vectors not having a finite e -orbit. The remaining (not shaded) part is our $V(n, \lambda_2)$, with its $\mathfrak{g}$ -types clearly visible.

From Corollary 8, Theorem 11 and Corollary 26 we have the following consequence:

Corollary 27. *For $\lambda_2 \neq 0$ we have $V(0, \lambda_2) \cong V(0, -\lambda_2) \cong Q(0, \lambda_2^2)$.*

From Lemma 16, Proposition 21, and the definition of $V(n, \lambda_2)$, we have:

Proposition 28. *For $n \in \mathbb{Z}$, $\lambda_2 \neq 0$ and $\mu \in \mathbb{Z}_{\geq 0}$, we have the following isomorphism of $\mathfrak{T}$ -modules:*

$$V(n, \lambda_2) \otimes L(\mu) \cong V(n + \mu, \lambda_2) \oplus V(n + \mu - 2, \lambda_2) \oplus \dots \oplus V(n - \mu, \lambda_2).$$

Now we can completely describe the universal modules:

Proposition 29. *For $n \in \mathbb{Z}_{\geq 0}$ and $\chi \neq 0$, choose any square root λ_2 of χ . Then*

$$Q(n, \chi) \cong V(n, \lambda_2) \oplus V(n - 2, \lambda_2) \oplus \dots \oplus V(-n, \lambda_2).$$

Moreover, $V(n, \lambda_2) \cong V(-n, -\lambda_2)$.

Proof. The first claim follows from Proposition 28, Corollary 27 and (5). The second claim follows from the first one by comparing both choices $\pm\lambda_2$ and central characters of the summands. $\square$

Theorem 30. *Let V be a simple $\mathfrak{g}$ -Harish-Chandra module for $\mathfrak{T}$. Denote by $\chi = \chi(\bar{C})$ the purely Takiff part of the central character, and suppose $L(n)$, $n \in \mathbb{Z}_{\geq 0}$, is the minimal $\mathfrak{g}$ -type of V .*

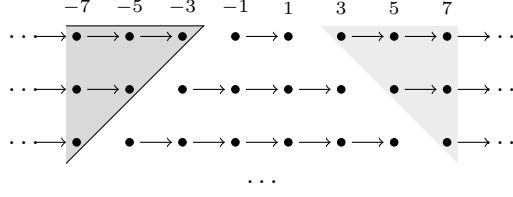
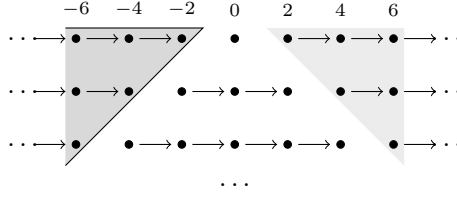
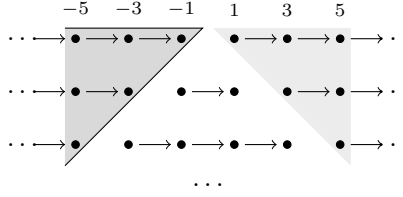
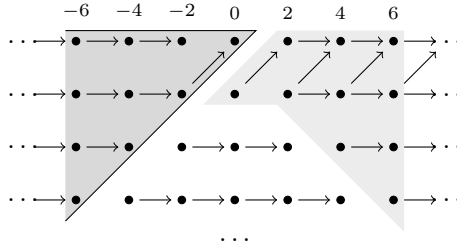
- *If $\chi \neq 0$, then $V \cong V(n, \lambda_2)$, for a square root λ_2 of χ .*
- *If $\chi = 0$, then $V \cong L(n)$ with the trivial $\bar{\mathfrak{g}}$ -action.*

In other words, $V(n, \lambda_2)$, $n \in \mathbb{Z}$, $\lambda_2 \in \mathbb{C} \setminus \{0\}$, together with the finite-dimensional simple $\mathfrak{g}$ -modules constitute a complete list of simple $\mathfrak{g}$ -Harish-Chandra modules for $\mathfrak{T}$. The only isomorphisms between different members of the list are $V(n, \lambda_2) \cong V(-n, -\lambda_2)$.

Proof. By Proposition 7(b), V is a quotient of $Q(n, \chi)$.

If $\chi \neq 0$, from Proposition 29 and Corollary 26 we see that the only possible choices with the correct minimal $\mathfrak{g}$ -type are $V(n, \lambda_2)$ or $V(n, -\lambda_2)$.

If $\chi = 0$, by the second part of Theorem 11, we see that the only possible simple quotients of $Q(n, 0) \cong Q(0, 0) \otimes L(n)$ are just finite-dimensional simple $\mathfrak{g}$ -modules with the trivial $\bar{\mathfrak{g}}$ -action. $\square$

FIGURE 1. $V(-1, \lambda_2) = \mathbf{EA}(\Delta(-3, \lambda_2))$ FIGURE 2. $V(0, \lambda_2) = \mathbf{EA}(\Delta(-2, \lambda_2))$ FIGURE 3. $V(1, \lambda_2) = \mathbf{EA}(\Delta(-1, \lambda_2))$ FIGURE 4. $V(2, \lambda_2) = \mathbf{EA}(\Delta(0, \lambda_2))$

4.6. Extensions. Here we calculate the first extension groups of simple $\mathfrak{g}$ -Harish-Chandra modules, restricting to the infinite-dimensional cases, i.e., a non-trivial central character. Since in that case non-isomorphic $\mathfrak{g}$ -Harish-Chandra modules have different central characters, there are no non-trivial extensions between them. So it only makes sense to calculate the self-extensions.

Theorem 31. *For an infinite-dimensional simple $\mathfrak{g}$ -Harish-Chandra module V , we have*

$$\mathrm{Ext}^1(V, V) \cong \mathbb{C}.$$

Proof. Assume first that $V = Q(0, \chi)$ for $\chi \neq 0$, and suppose we have a non-split short exact sequence $0 \rightarrow V \xrightarrow{i} M \xrightarrow{p} V \rightarrow 0$. Denote by $1 \in V$ the generator from

$L(0)$, set $w = i(1) \in M$ and find $v \in M$ such that $p(v) = 1$. Since the sequence must split in the category of $\mathfrak{g}$ -modules, v generates the trivial $\mathfrak{g}$ -submodule in M . By the universal property, there is a $\mathfrak{T}$ -homomorphism $f: Q(0) \rightarrow M$, and the triangle below commutes:

$$\begin{array}{ccccccc} & & Q(0) & & & & \\ & & \downarrow f & \searrow & & & \\ 0 & \longrightarrow & V^c & \longrightarrow & M & \longrightarrow & V \longrightarrow 0. \end{array}$$

The map f must be surjective, since otherwise its image would define a splitting of the short exact sequence. So, there is an element in $Q(0)$ that maps to w via f ; by Lemma 10 and Proposition 6 such an element is necessarily of the form $p(\bar{C})$ for some polynomial p . Since the triangle above commutes, we must have $p(\chi) = 0$. Since $[M: L(0)] = 2$, we can take $p(\bar{C}) = \bar{C} - \chi$. From this one can see that $\text{Ker } f$ is generated by $(\bar{C} - \chi)^2$, i.e., $M \cong U(\bar{\mathfrak{g}})/(\bar{C} - \chi)^2$. This uniquely determines M . Conversely, one sees directly that such M defines a non-split self-extension of $Q(0, \chi)$.

The general statement is obtained from this by translation functors, i.e., tensoring extensions of $Q(0, \chi)$ by $L(n)$ and then taking the component with the correct central character (see Proposition 28). This functor defines a homomorphism of abelian groups $\text{Ext}^1(V(0, \lambda_2), V(0, \lambda_2)) \rightarrow \text{Ext}^1(V(n, \lambda_2), V(n, \lambda_2))$. In the same way we get a homomorphism in the other direction.

The fact that these homomorphisms compose to the identities on the Ext^1 groups is an easy application of the 5-lemma. $\square$

4.7. Annihilators. We will prove here that the infinite-dimensional simple $\mathfrak{g}$ -Harish-Chandra modules have the same annihilators in $U = U(\mathfrak{T})$ as the corresponding Verma modules. We start by showing that, in the cases we are interested in, the localization does not decrease the annihilator. Then we construct a certain “inverse” of the functor $\mathbf{EA}$, which will produce Verma modules out of $\mathfrak{g}$ -Harish-Chandra modules. This will be given by the localization by $\bar{e}$, i.e. tensoring with $S_{\bar{e}}$ over U .

Lemma 32. *Let x be an ad-nilpotent element in $\mathfrak{T}$, and M a $\mathfrak{T}$ -module on which x acts injectively. Then in U we have*

$$\text{Ann}(M) = \text{Ann}\left(U_{(x)} \otimes_U M\right) \subseteq \text{Ann}\left(S_x \otimes_U M\right).$$

Proof. The only non-obvious thing to prove is if $u \in \text{Ann}(M)$, then $ux^{-n} \otimes m = 0$, for all $n \geq 1$ and $m \in M$.

By assumption, for any $u \in U$ there exists $k_1 > 0$ such that

$$0 = (\text{ad}(x))^{k_1}(u) = \sum_{i=0}^{k_1} (-1)^i \binom{k_1}{i} x^i u x^{k_1-i},$$

so $x^{k_1}u = u_1x$ for $u_1 := \sum_{i=0}^{k_1-1} (-1)^{k_1+i} \binom{k_1}{i} x^i u x^{k_1-i-1}$. If $u \in \text{Ann}(M)$, then so is u_1 too, since $\text{Ann}(M)$ is a two-sided ideal. We can inductively apply the same procedure on u_1 to get k_2 such that $x^{k_2}u_1 = u_2x$, etc. Repeating this n times, we get $x^{k_n}u_{n-1} = u_n$ for some $u_n \in \text{Ann}(M)$. From the construction it follows that

$$x^{k_n+\dots+k_1} \cdot ux^{-n} \otimes m = u_n \otimes m = 1 \otimes u_n m = 0.$$

Since x acts injectively on M , the same is true for $U_{(x)} \otimes_U M$, so we conclude that $ux^{-n} \otimes m = 0$. $\square$

Proposition 33. *Suppose $n \in \mathbb{Z}$ and $\lambda_2 \neq 0$.*

- (a) *The element $\bar{e}$ acts injectively on $V(n, \lambda_2)$.*
 (b) *The module $S_{\bar{e}} \otimes_U V(n, \lambda_2)$ is isomorphic to the direct sum of Verma modules $\Delta(n-2, \lambda_2) \oplus \Delta(-n-2, -\lambda_2)$.*

Proof. The first claim follows from Lemma 10(b) for $V(0, \lambda_2) \cong Q(0, \lambda_2^2)$, and from Proposition 29 for general $V(n, \lambda_2)$.

The second claim we also prove first for $n = 0$, and again translate the result to the other cases. From Lemma 10(b), we get a basis for $W := S_{\bar{e}} \otimes_U Q(0, \lambda_2^2)$ consisting of

$$(\bar{e})^{-k} \bar{f}^l \bar{h}^\epsilon, \text{ for } k > 0, l \geq 0, \epsilon \in \{0, 1\},$$

where $\mathfrak{g}$ acts by the adjoint action, and $\bar{\mathfrak{g}}$ by the (commutative) multiplication. We denote this action of $\mathfrak{T}$ by $\circ$.

Consider the following two elements in W :

$$w_{\pm} := (\bar{e})^{-1} \pm \frac{1}{\lambda_2} (\bar{e})^{-1} \bar{h}.$$

It is an easy calculation to see that $e \circ w_{\pm} = \bar{e} \circ w_{\pm} = 0$, $h \circ w_{\pm} = -2w_{\pm}$, and $\bar{h} \circ w_{\pm} = \pm \lambda_2 w_{\pm}$, from which it follows by the universal property of Verma modules that each $w_{\pm}$ generates a copy of $\Delta(-2, \pm \lambda_2)$ in W . Because of their simplicity, these submodules can only intersect trivially. By comparing the dimensions of weight spaces, we conclude that W cannot have any other composition factor, i.e.,

$$(14) \quad W \cong \Delta(-2, \lambda_2) \oplus \Delta(-2, -\lambda_2).$$

In general, we calculate $S_{\bar{e}} \otimes_U Q(n, \lambda_2^2)$ in two ways and compare the results:

$$\begin{aligned} S_{\bar{e}} \otimes_U Q(n, \lambda_2^2) &\cong S_{\bar{e}} \otimes_U \left(Q(0, \lambda_2^2) \otimes L(n) \right) && \text{by (5),} \\ &\cong \left(S_{\bar{e}} \otimes_U Q(0, \lambda_2^2) \right) \otimes L(n) && \text{by Lemma 18,} \\ &\cong (\Delta(-2, \lambda_2) \oplus \Delta(-2, -\lambda_2)) \otimes L(n) && \text{by (14),} \\ (15) \quad &\cong \bigoplus_{k=0}^n \Delta(n-2-2k, \lambda_2) \oplus \bigoplus_{k=0}^n \Delta(n-2-2k, -\lambda_2) && \text{by Lemma 16.} \end{aligned}$$

On the other hand, by Proposition 29 we have

$$(16) \quad S_{\bar{e}} \otimes_U Q(n, \lambda_2^2) \cong \bigoplus_{k=0}^n S_{\bar{e}} \otimes_U V(n-2k, \lambda_2).$$

By comparing the central characters (which are preserved under the localization) of the direct summands in (15) and (16), the claim (b) follows. $\square$

From Lemma 32, Proposition 33 and the definition of $V(n, \lambda_2)$ we have:

Corollary 34. *Suppose $n \in \mathbb{Z}$ and $\lambda_2 \neq 0$. Then*

$$\text{Ann}(V(n, \lambda_2)) = \text{Ann}(\Delta(n-2, \lambda_2)) = \text{Ann}(\Delta(-n-2, -\lambda_2)).$$

We want to prove that these annihilators are centrally generated. It is easier to do this for Verma modules. This has already been proved in [BL17, Proposition 6.1]. We present a different and a more direct proof, and along the way reveal some structure of the quotients of U by the centrally generated ideals.

For this, we need to express elements of U modulo a maximal ideal in the center in a convenient way. This we describe in the next two lemmas. We denote by $U_0 := U(\mathfrak{T})_0$, the zero-weight space of $\mathfrak{h}$ in U .

Lemma 35. *The subalgebra U_0 of U is generated by $S = \{h, \bar{h}, fe, \bar{f}e, f\bar{e}, \bar{f}\bar{e}\}$.*

Proof. We need to prove that any product $x = x_1 x_2 \dots x_k$, where each x_i belongs to the standard basis of $\mathfrak{T}$, with the property that the number of i 's for which $x_i \in \{f, \bar{f}\}$ is equal to the number of j 's for which $x_j \in \{e, \bar{e}\}$, can be generated by elements in S . We prove this by induction on k . If $x_1 x_2 \dots x_k$ consists only of h and $\bar{h}$, we are done. If not, chose some $x_i \in \{f, \bar{f}\}$ and $x_j \in \{e, \bar{e}\}$, and assume without loss of generality $i < j$. We commute them to the right-most place:

$$x = \underbrace{(x_1 \dots \hat{x}_i \dots \hat{x}_j \dots x_k)}_{x'} \underbrace{(x_i x_j)}_{\in S} + \sum_t y_t,$$

where the factors with hat are omitted. It is clear from the commutation relations that x' and all y_t are products of the basis elements with the same property, but shorter. We are done by induction. $\square$

Lemma 36. *Fix an algebra homomorphism $\chi: Z(\mathfrak{T}) \rightarrow \mathbb{C}$. For any $u \in U_0 / U_0 \cdot \text{Ker } \chi$ there exists $n \in \mathbb{Z}_{\geq 0}$ such that $(\bar{f}\bar{e})^n \cdot u$ is equal to a linear combination of monomials of the form $h^k (f\bar{e})^l (\bar{h})^m$ for $k, l, m \in \mathbb{Z}_{\geq 0}$, modulo $U_0 \cdot \text{Ker } \chi$.*

Proof. In the quotient above, by using (7) we can express $\bar{f}\bar{e}$ as a linear combination of $\bar{h}^2$ and 1, and also $\bar{f}e$ as a linear combination of $\bar{h}h$, $\bar{h}$, $f\bar{e}$ and 1. Using this with Lemma 10, we see that the generators in the quotient are just h , $\bar{h}$, fe and $f\bar{e}$.

First let us assume that $u = x_1 x_2 \dots x_r$, a product of these four generators in any order. Since h commutes with everything here, we can ignore it. Denote by $a = fe$, $b = f\bar{e}$, $c = \bar{h}$, $\chi_1 = \chi(h)$ and $\chi_2 = \chi(\bar{h})$. One can check that we have the following relations in the quotient:

$$(17) \quad [b, a] = ac - hb,$$

$$(18) \quad [c, a] = 4b + hc + 2c - \chi_1,$$

$$(19) \quad [c, b] = \frac{1}{2}c^2 - \frac{1}{2}\chi_2,$$

$$(20) \quad \bar{f}\bar{e} \cdot a = -b^2 - hc^2 - \frac{3}{2}c^2 - hbc - 2bc + \frac{\chi_1}{2}b + \frac{\chi_1}{2}c + \frac{\chi_2}{2}.$$

Suppose that $x_1 \neq a$, but some $x_i = a$, and assume i is minimal. Using the relations (17) and (18), we commute $x_{i-1}x_i = x_i x_{i-1} + [x_{i-1}, x_i]$. This way u becomes a sum of several monomials, each of which has either one a less, or have their most-left a one place closer to the most left position. It follows that we can move this a to the most left part in a finite number of steps, i.e., we can express u as a finite sum $u = \sum a y_t + \sum w_t$, where y_t is a finite product of a 's, b 's and c 's, but has at least one a less than the original expression of u had, and w_t is a finite product of b 's and c 's.

From (20) and the relation $4\bar{f}\bar{e} = \chi_2 - c^2$, it follows that $\bar{f}\bar{e} \cdot u$ is a finite sum $\sum z_t$, where each z_t is a product of a 's, b 's and c 's, but has at least one a less than the

original expression of u had. By induction, for some k we get that $(\bar{f}\bar{e})^k \cdot u$ is a finite sum of products of b 's and c 's.

Now observe that any product of b 's and c 's can be expressed as a linear combination of standard monomials $b^i c^j$, using the relation (19) and a very similar reasoning as before. The point is that a does not appear in $[c, b]$ in (19), so we will not end up in an infinite loop.

Finally, note that the argument is essentially the same if we started from u equal to a linear combination of products of the generators, instead of just one monomial. $\square$

Theorem 37. *Suppose $n \in \mathbb{Z}$ and $\lambda_2 \neq 0$. The annihilators in Corollary 34 are centrally generated. More precisely, they are equal to the ideal $U \cdot \text{Ker } \chi$, where*

$$\chi: Z(\mathfrak{T}) = \mathbb{C}[C, \bar{C}] \rightarrow \mathbb{C}$$

is a homomorphism of algebras defined on the generators by $C \mapsto n\lambda_2$ and $\bar{C} \mapsto \lambda_2^2$.

Proof. We prove this for the annihilator of the Verma module $\Delta := \Delta(n-2, \lambda_2)$. This is known from [BL17, Proposition 6.1], but we present here a different and a more direct proof.

The inclusion $U \cdot \text{Ker } \chi \subseteq \text{Ann}(\Delta)$ is trivial. For the converse, recall that $\text{Ann}(\Delta)$ is stable under the adjoint action, so it is generated by its U_0 part. So, it is enough to prove

$$U_0 \cap \text{Ann}(\Delta) \subseteq U_0 \cdot \text{Ker } \chi.$$

To prove this, for any non-zero element $u \in U_0 / U_0 \cdot \text{Ker } \chi$ we want to find an element from Δ which is not annihilated by u . Because of Lemma 36, we can assume without loss of generality that

$$u = \sum_{k,l,m \geq 0} \alpha_{klm} h^k (\bar{f}\bar{e})^l (\bar{h})^m,$$

with $\alpha_{klm} \in \mathbb{C}$ and only finitely many non-zero. Define a polynomial (with commutative variables) by the same scalars: $p(x, y, z) = \sum_{k,l,m \geq 0} \alpha_{klm} x^k y^l z^m \in \mathbb{C}[x, y, z]$.

Denote by v the highest weight vector in Δ , by Δ_q the weight space in Δ of weight $n-2-2q$, $q \geq 0$, and recall that it has basis $f^i \bar{f}^{q-i} v$ for $i = 0, 1, \dots, q$. Similarly to (8), one can prove the following formulas for the action on Δ :

$$\begin{aligned} h \cdot f^i \bar{f}^{q-i} v &= (n-2-2q) f^i \bar{f}^{q-i} v, \\ \bar{e} \cdot f^i \bar{f}^{q-i} v &= i\lambda_2 f^i \bar{f}^{q-i} v - i(i-1) f^{i-1} \bar{f}^{q-i+1} v, \\ \bar{h} \cdot f^i \bar{f}^{q-i} v &= \lambda_2 f^i \bar{f}^{q-i} v - 2i f^{i-1} \bar{f}^{q-i+1} v. \end{aligned} \tag{21}$$

It follows that in this basis of Δ_q , the operator representing u is upper-triangular, with the diagonal entries $p(n-2-2q, i\lambda_2, \lambda_2)$, $i = 0, \dots, q$. We would like to find a basis element $f^i \bar{f}^{q-i} v$, for which $p(n-2-2q, i\lambda_2, \lambda_2) \neq 0$. However, a problem arises if $p(x, y, z)$ is divisible by $(z - \lambda_2)$.

We claim that we can decompose

$$(22) \quad p(x, y, z) = \tilde{p}(x, y, z) \cdot (z - \lambda_2)^r$$

for some $r \geq 0$, such that $\tilde{p}(n-2-2q, i\lambda_2, \lambda_2)$ is not identically zero for $(q, i) \in D$, where $D \subseteq \mathbb{C}^2$ is any Zariski dense subset.

To prove this claim, write $p(x, y, z) = \sum_{j=0}^m p_j(x, y)(z - \lambda_2)^j$. Suppose that this is zero when evaluated on $D \times \{\lambda_2\}$ for a Zariski dense set $D \subseteq \mathbb{C}^2$. It follows that $p_0(x, y) = 0$ (on $\mathbb{C}^2$), so $p(x, y, z) = p^{(1)}(x, y, z)(z - \lambda_2)$, for a polynomial $p^{(1)}(x, y, z)$ of a strictly smaller total degree. If necessary, we continue to apply the same argument inductively

on $p^{(1)}(x, y, z)$, etc., until we reach (22) with $\tilde{p}(x, y, \lambda_2)$ non-zero on some point in $(x, y) \in D$. The number r is independent of D , since the set $\{(x, y) : \tilde{p}(x, y, \lambda_2) \neq 0\}$ is non-empty and Zariski open, hence intersects any Zariski dense set in $\mathbb{C}^2$.

The claim is now proved, because the map $(q, i) \mapsto (n - 2 - 2q, i\lambda_2)$ is an algebraic isomorphism $\mathbb{C}^2 \rightarrow \mathbb{C}^2$. Here it is crucial that $\lambda_2 \neq 0$.

Write $\tilde{p}(x, y, z) = \sum_{k,l,m \geq 0} \tilde{\alpha}_{klm} x^k y^l z^m$, and define $\tilde{u} = \sum_{k,l,m \geq 0} \tilde{\alpha}_{klm} h^k (f\bar{e})^l (\bar{h})^m$. Then it is also true that

$$u = \tilde{u} \cdot (\bar{h} - \lambda_2)^r,$$

since the monomials in u and $\tilde{u}$ have $\bar{h}$ on the most-right position, so no commuting of the variables is necessary.

There exists a pair (q, i) from the cone $\{(q, i) \in \mathbb{Z} \times \mathbb{Z} : q \geq r, 0 \leq i \leq q - r\}$ (which is Zariski dense in $\mathbb{C}^2$), such that $\tilde{p}(n - 2 - 2q, i\lambda_2, \lambda_2) \neq 0$. Put $w := f^{i+r} \bar{f}^{q-i-r} v \in \Delta_q$. It follows from (21) that $(\bar{h} - \lambda_2)^r \cdot w = c \cdot f^i \bar{f}^{q-i}$, for some constant $c \neq 0$. From this we have that

$$\begin{aligned} u \cdot w &= c \cdot \tilde{u} \cdot f^i \bar{f}^{q-i} v \\ &= c \cdot \tilde{p}(n - 2 - 2q, i\lambda_2, \lambda_2) \cdot f^i \bar{f}^{q-i} v + \sum_{j=0}^{i-1} c_j f^j \bar{f}^{q-j} v \neq 0. \end{aligned} \quad \square$$

4.8. The action of finite-dimensional $\mathfrak{sl}_2$ -modules. Denote by $\mathcal{F}$ the monoidal category of finite-dimensional $\mathfrak{sl}_2$ -modules. For a fixed non-zero $\chi \in \mathbb{C}$, denote by $\mathcal{H}_\chi$ the category of semi-simple $\mathfrak{g}$ -Harish-Chandra $\mathfrak{T}$ -modules on which the action of the purely Takiff part of the center is given by χ .

Proposition 38. *For each non-zero χ , the category $\mathcal{H}_\chi$ is a simple module category over $\mathcal{F}$.*

Proof. The fact that $\mathcal{H}_\chi$ is a module category over $\mathcal{F}$ follows directly from Proposition 28. Since $\mathcal{H}_\chi$ is semi-simple by definition, to show that it is a simple module category over $\mathcal{F}$ it is enough to show that, starting from any simple object of $\mathcal{H}_\chi$ and tensoring it with finite-dimensional $\mathfrak{sl}_2$ -modules, we can obtain any other simple object of $\mathcal{H}_\chi$ as a direct summand, up to isomorphism. This claim follows by combining Proposition 28 with Theorem 30. $\square$

We note that, by Proposition 28, the combinatorics of the $\mathcal{F}$ -module category $\mathcal{H}_\chi$ does not depend on χ .

5. $\mathfrak{sl}_2$ -HARISH-CHANDRA MODULES FOR THE SCHRÖDINGER LIE ALGEBRA

5.1. Setup. The *Schrödinger Lie algebra* $\mathfrak{s}$ can be defined by basis $\{e, h, f, p, q, z\}$ and the following relations: in addition to the usual $\mathfrak{g} := \mathfrak{sl}_2$ relations on e, h, f , we also have

$$\begin{aligned} [e, p] &= 0, & [h, p] &= p, & [f, p] &= q, \\ [e, q] &= p, & [h, q] &= -q, & [f, q] &= 0, & [p, q] &= z. \end{aligned}$$

and z is declared to commute with all $\mathfrak{s}$. It is clear that $\mathfrak{s} = \mathfrak{g} \ltimes \mathfrak{h}$, where $\mathfrak{h}$ is the ideal spanned by p, q, z , and is isomorphic to the 3-dimensional Heisenberg Lie algebra. As a $\mathfrak{g}$ -module, $\mathfrak{h}$ is isomorphic to $L(1) \oplus L(0)$.

The nilradical of $\mathfrak{s}$ is $\text{Nrad}(\mathfrak{s}) = [\mathfrak{s}, \mathfrak{h}] = \mathfrak{h}$. Recall that this means that $\mathfrak{h}$ must necessarily annihilate any simple finite-dimensional $\mathfrak{s}$ -module.

There is also the *centerless Schrödinger Lie algebra* $\bar{\mathfrak{s}} := \mathfrak{s}/\mathbb{C}z$, which is isomorphic to the semi-direct product $\mathfrak{g} \ltimes L(1)$.

The disclaimer from the previous section related to [BL17] applies to the present section with respect to [BL18].

The algebra $U(\mathfrak{s})$ is free as a module over its center $Z(\mathfrak{s})$, and $Z(\mathfrak{s})$ is generated by two algebraically independent generators (see e.g. [DLMZ14]):

$$(23) \quad C := (h^2 + h + 4fe)z - 2(fp^2 - eq^2 - hpq), \text{ and } z.$$

It is also clear that $Z(\mathfrak{s}) \cap U(\mathfrak{h}) = \mathbb{C}[z]$, which we will refer to as the *purely Schrödinger part of the center*. For a module with central character, the scalar by which z acts is usually called the *central charge* of the module.

The theory we develop here for the Schrödinger Lie algebra is very similar to the Takiff $\mathfrak{sl}_2$ case. So we will omit most of the details, as they are usually analogous, but easier. One reason for this is that the purely Schrödinger part of the center is generated by a degree 1 element, and for the Takiff $\mathfrak{sl}_2$ we had a degree 2 element. However, a small complication now is that the radical of $\mathfrak{s}$ is not abelian anymore.

5.2. Universal modules. As before, the universal modules are induced from $\mathfrak{g}$, i.e., for $n \in \mathbb{Z}_{\geq 0}$ set $Q(n) := \text{Ind}_{\mathfrak{g}}^{\mathfrak{s}} L(n) = U(\mathfrak{s}) \otimes_{U(\mathfrak{g})} L(n) \cong U(\mathfrak{h}) \otimes_{\mathbb{C}} L(n)$. Recall that $Q(n) \cong Q(0) \otimes L(n)$, where we consider $L(n)$ as an $\mathfrak{s}$ -module with the trivial $\mathfrak{h}$ -action.

Proposition 39. *We have the following isomorphisms of algebras:*

$$\text{End}(Q(0)) \cong U(\mathfrak{h})^{\mathfrak{g}} = \mathbb{C}[z],$$

where $U(\mathfrak{h})^{\mathfrak{g}}$ denotes the invariants of the adjoint action of $\mathfrak{g}$ on $U(\mathfrak{h})$.

Proof. The isomorphism $\text{End}(Q(0))^{\text{op}} \cong U(\mathfrak{h})^{\mathfrak{g}}$ follows from the same argument as in the proof of Proposition 6. The inclusion $U(\mathfrak{h})^{\mathfrak{g}} \supseteq \mathbb{C}[z]$ is obvious. The converse follows easily from the following commutation relations, which can be proved e.g. by induction:

$$(24) \quad \begin{aligned} [h, p^m q^n] &= (m - n)p^m q^n, \\ [e, p^m q^n] &= np^{m+1}q^{n-1} - \frac{n(n-1)}{2}p^m q^{n-2}z, \\ [f, p^m q^n] &= mp^{m-1}q^{n+1} - \frac{m(m-1)}{2}p^{m-2}q^n z. \end{aligned} \quad \square$$

Fix $\chi \in \mathbb{C}$, and denote by $\mathfrak{m}_{\chi}$ the maximal ideal $(z - \chi) \subseteq \mathbb{C}[z]$. As before, we define the *universal module* as $Q(n, \chi) := Q(n) / \mathfrak{m}_{\chi} Q(n)$. It clearly has central charge χ . As before, we have $Q(n, \chi) \cong Q(0, \chi) \otimes L(n)$.

Lemma 40. (a) *As $\mathfrak{s}$ -modules, $Q(0) \cong U(\mathfrak{h})$ and $Q(0, \chi) \cong U(\mathfrak{h}) / (z - \chi)$, where $\mathfrak{g}$ acts by the adjoint action, and $\mathfrak{h}$ by the left multiplication. The set $\{p^i q^j : i, j \geq 0\}$ is a basis for $Q(0, \chi)$.*

(b) *As a $\mathfrak{g}$ -module, $Q(0, \chi) \cong \bigoplus_{k \geq 0} L(k)$. Moreover, p^k is the highest weight vector in $L(k)$.*

(c) *C acts as zero on $Q(0)$ and every $Q(0, \chi)$.*

Proof. The first claim is clear. We use it to prove the others.

For the second claim, note that p^k generates a $\mathfrak{g}$ -submodule isomorphic to $L(k)$. Since the action of $\mathfrak{g}$ preserves $Q^n := \text{span}\{p^i q^j : i + j \leq n\}$, by counting dimensions we see that $Q^n \cong \bigoplus_{k=0}^n L(k)$. The claim now follows by taking colimits.

The last claim can be checked directly (enough on the generator of $Q(0)$). $\square$

From the previous lemma, the ClebschGordan coefficients for $\mathfrak{sl}_2$, and the adjunction, the following is not hard to deduce:

Proposition 41. (a) $Q(n, \chi)$ is a $\mathfrak{g}$ -Harish-Chandra module, and for $k \geq 0$:

$$(25) \quad [Q(n, \chi) : L(k)] = \min\{k + 1, n + 1\}.$$

(b) Let V be any simple $\mathfrak{s}$ -module that has some $L(n)$ as a simple $\mathfrak{g}$ -submodule. Then V is a quotient of $Q(n, \chi)$ for a unique χ . In particular, V is a $\mathfrak{g}$ -Harish-Chandra module, and (25) gives an upper bound for the multiplicities of its $\mathfrak{g}$ -types.

(c) For a fixed χ , there exists a unique simple $\mathfrak{s}$ -module which contains $L(0)$ as a $\mathfrak{g}$ -submodule and has central charge χ . Moreover, it is a $\mathfrak{g}$ -Harish-Chandra module.

Theorem 42. The module $Q(0, \chi)$ is simple if and only if $\chi \neq 0$.

The module $Q(0, 0)$ has infinite length, and an $\mathfrak{s}$ -filtration whose composition factors are $L(0), L(1), L(2) \dots$ with the trivial action of $\mathfrak{h}$.

Proof. As before, the $\mathfrak{s}$ -action on $Q(0)$ and $Q(0, \chi)$ will be denoted by $\circ$.

Note that $[q, p^n] = -np^{n-1}z$ and $[p, q^n] = nq^{n-1}z$. Using this and the equations (24), one can check that

$$(pf - nq) \circ p^n = \frac{n(n+1)}{2} \chi \cdot p^{n-1}.$$

So if $\chi \neq 0$, the module $Q(0, \chi)$ is simple.

Alternatively, one can use a nilradical argument analogous to the one in Remark 12.

If $\chi = 0$, then p and q commute in $Q(0, \chi)$, and $\mathfrak{g}$ preserves the total degree of monomials $p^i q^j$. The rest of the proof is obvious. $\square$

5.3. Verma modules. Verma modules for the Schrödinger Lie algebra are studied in detail in [DLMZ14].

In the triangular decomposition (1) we have

$$\tilde{\mathfrak{n}}_- = \text{span}\{f, q\}, \quad \tilde{\mathfrak{h}} = \text{span}\{h, z\}, \quad \text{and} \quad \tilde{\mathfrak{n}}_+ = \text{span}\{e, p\}.$$

For an element $\lambda \in \tilde{\mathfrak{h}}^*$, denote $\lambda_1 := \lambda(h)$ and $\lambda_2 := \lambda(z)$.

Proposition 43 ([DLMZ14, Proposition 5]). If $\lambda_2 \neq 0$, then the Verma module $\Delta(\lambda)$ is simple for any $\lambda_1 \in \mathbb{Z}$.

It is easy to see that the central element C acts on the Verma module $\Delta(\lambda)$ as the scalar $(\lambda_1 + 1)(\lambda_1 + 2)\lambda_2$, and the central charge is $\chi := \lambda_2$, see (23). We will be concerned mostly with non-zero central charge cases.

Lemma 44. Non-isomorphic Verma modules $\Delta(\lambda)$ and $\Delta(\lambda')$ with the same non-zero central charge have the same central character if and only if $\lambda'_1 = -\lambda_1 - 3$.

Proof. This reduces to solving the equation $(\lambda_1 + 1)(\lambda_1 + 2) = (\lambda'_1 + 1)(\lambda'_1 + 2)$. $\square$

Lemma 45. *As a $\mathfrak{g}$ -module, $\Delta(\lambda)$ has a filtration with subquotients isomorphic to the $\mathfrak{g}$ -Verma modules $\Delta^{\mathfrak{g}}(\lambda_1 - k)$, $k = 0, 1, 2, \dots$*

If $\lambda_2 = 0$ or $\lambda_1 \notin \mathbb{Z}_{\geq 0}$, then as a $\mathfrak{g}$ -module we have $\Delta(\lambda) \cong \bigoplus_{k \geq 0} \Delta^{\mathfrak{g}}(\lambda_1 - k)$. Otherwise ($\lambda_2 \neq 0$ and $\lambda_1 \in \mathbb{Z}_{\geq 0}$) we have as $\mathfrak{g}$ -modules

$$\Delta(\lambda) \cong \Delta^{\mathfrak{g}}(-1) \oplus \bigoplus_{k=2}^{\lambda_1+2} P^{\mathfrak{g}}(-k) \oplus \bigoplus_{k \geq 3} \Delta^{\mathfrak{g}}(-\lambda_1 - k).$$

Proof. Denote by v_λ a basis element of $\mathbb{C}_\lambda$. Then $\Delta(\lambda)$ has a basis of weight vectors $\{f^i q^j v_\lambda : i, j \geq 0\}$. A direct computation shows that

$$e \cdot f^i q^j v_\lambda = i(\lambda_1 - i - j + 1) f^{i-1} q^j v_\lambda + \lambda_2 \frac{j(j-1)}{2} f^i q^{j-2} v_\lambda.$$

This implies that the required filtration is given by the degree of q . The subquotients are given by the span of $\{f^i q^k v_\lambda : i \geq 0\}$, which is clearly isomorphic to $\Delta^{\mathfrak{g}}(\lambda_1 - k)$. The rest can be proved in the same way as for Lemma 15. $\square$

Lemma 46. *For $\lambda_1 \in \mathbb{Z}$, $\lambda_2 \neq 0$ and $\mu \in \mathbb{Z}_{\geq 0}$ there is an isomorphism of $\mathfrak{s}$ -modules*

$$\Delta(\lambda) \otimes L(\mu) \cong \Delta(\lambda_1 + \mu, \lambda_2) \oplus \Delta(\lambda_1 + \mu - 2, \lambda_2) \oplus \dots \oplus \Delta(\lambda_1 - \mu, \lambda_2).$$

Proof. The left-hand side has a filtration with subquotients equal to the summands on the right-hand side. But these subquotients have different central characters by Lemma 44, since the first components of their highest weights have the same parity, so they must split. $\square$

5.4. Enright-Arkhipov completion. Fix an ad-nilpotent element $x \in \mathfrak{s}$ (for example f or p , which we will use), and denote by $S_x := U(\mathfrak{s})_{(x)} / U(\mathfrak{s})$ the localization of the algebra $U(\mathfrak{s})$ by x , modulo the canonical copy of $U(\mathfrak{s})$ inside it. This is a $U(\mathfrak{s})$ -bimodule. For an $\mathfrak{s}$ -module M write

$$\mathbf{EA}(M) := {}^e \left(S_f \otimes_{U(\mathfrak{s})} M \right).$$

As before, one can check that this is a well defined functor on the category of $\mathfrak{s}$ -modules. Moreover, Proposition 21 is valid here, with the same proof.

Theorem 47. *Take λ with $\lambda_1 \in \mathbb{Z}$ and $\lambda_2 \neq 0$. Then $\mathbf{EA}(\Delta(\lambda))$ is a simple $\mathfrak{g}$ -Harish-Chandra module, and decomposes as a $\mathfrak{g}$ -module as*

$$(26) \quad \mathbf{EA}(\Delta(\lambda)) \cong \bigoplus_{k \geq 0} L \left(\left\lfloor \lambda_1 + \frac{3}{2} \right\rfloor - \frac{1}{2} + k \right).$$

Proof. Lemma 45, the fact that the functor $\mathbf{EA}$ commutes with the forgetful functor from $\mathfrak{s}$ -modules to $\mathfrak{g}$ -modules, and (the Schrödinger analogue of) Example 22 imply the decomposition (26).

Note that the lowest weight vector of a $L(\mu)$ in (26) is $f^{-1} q^t v_\lambda$, where $t := 2 + \lambda_1 + \mu$. To prove simplicity, it is enough for $\mu \geq \left\lfloor \lambda_1 + \frac{3}{2} \right\rfloor + \frac{1}{2}$ to find an element in $U(\mathfrak{s})$ that maps $f^{-1} q^t v_\lambda$ to $f^{-1} q^{t-1} v_\lambda$. One can check by direct calculation that

$$\left(p - \frac{1}{\mu} qe \right) \cdot f^{-1} q^t v_\lambda = \frac{\lambda_2 t}{2\mu} (\mu - \lambda_1 - 1) f^{-1} q^{t-1} v_\lambda.$$

The scalar on the right-hand side is non-zero because of the assumption on μ .

Alternatively, one can use a nilradical argument analogous to the one in Remark 24. $\square$

Theorem 47, Proposition 41(c) and Theorem 42 together give:

Corollary 48. *For $\lambda_2 \neq 0$ we have*

$$\mathbf{EA}(\Delta(-1, \lambda_2)) \cong \mathbf{EA}(\Delta(-2, \lambda_2)) \cong Q(0, \lambda_2).$$

5.5. Classification. The Enright-Arkhipov completion of Verma modules again gives us a family of $\mathfrak{g}$ -Harish-Chandra modules. This construction gives all infinite-dimensional $\mathfrak{g}$ -Harish-Chandra modules, as we will see in this subsection.

Definition 49. *For $n \in \mathbb{Z}$ and $\lambda_2 \neq 0$ denote*

$$V(n, \lambda_2) := \begin{cases} \mathbf{EA}(\Delta(n-1, \lambda_2)) & : n \geq 0, \\ \mathbf{EA}(\Delta(n-2, \lambda_2)) & : n \leq 0. \end{cases}$$

From Corollary 48 we have that $V(0, \lambda_2)$ is well-defined, and moreover isomorphic to $Q(0, \lambda_2)$. Note that the central element C acts on $V(n, \lambda_2)$ as $n(n+1)\lambda_2$ if $n \geq 0$, and as $n(n-1)\lambda_2$ if $n \leq 0$.

Theorem 47 and Lemma 44 easily give:

Corollary 50. *The module $V(n, \lambda_2)$ is a simple $\mathfrak{g}$ -Harish-Chandra module, and has $\mathfrak{g}$ -types $L(|n|), L(|n|+1), L(|n|+2) \dots$ with multiplicity one.*

If $V(n, \lambda_2) \cong V(n', \lambda'_2)$, then $\lambda'_2 = \lambda_2$ and $n' \in \{n, -n\}$.

On Figure 5 to 8 we present several $V(n, \lambda_2)$'s, and how they are constructed. It is interesting to compare this to the Takiff $\mathfrak{sl}_2$ case (cf. Figure 1 to 4).

Proposition 51. *For $n \in \mathbb{Z}_{\geq 0}$ and $\lambda_2 \neq 0$ we have $V(-n, \lambda_2) \cong V(n, \lambda_2)$. Moreover,*

$$Q(n, \lambda_2) \cong V(n, \lambda_2) \oplus V(n-1, \lambda_2) \oplus \dots \oplus V(0, \lambda_2).$$

Proof. We use induction over n . The basis is given in Corollary 48. Suppose the proposition is true for all $k = 0, \dots, n-1$ where $n \geq 1$ is fixed. Observe that, using (the Schrödinger version of) Proposition 21 and Lemma 46, we have

(27)

$$\begin{aligned} Q(n, \lambda_2) &\cong Q(0, \lambda_2) \otimes L(n) \cong \mathbf{EA}(\Delta(-1, \lambda_2)) \otimes L(n) \\ &\cong \mathbf{EA}(\Delta(n-1, \lambda_2)) \oplus \mathbf{EA}(\Delta(n-3, \lambda_2)) \oplus \dots \oplus \mathbf{EA}(\Delta(-n-1, \lambda_2)) \\ &\cong V(n, \lambda_2) \oplus V(n-2, \lambda_2) \oplus \dots \oplus V(\epsilon, \lambda_2) \oplus \\ &\quad \oplus V(\epsilon-1, \lambda_2) \oplus V(\epsilon-3, \lambda_2) \oplus \dots \oplus V(-n+1, \lambda_2), \end{aligned}$$

where $\epsilon \in \{0, 1\}$ is of the same parity as n . By inductive assumption it follows that

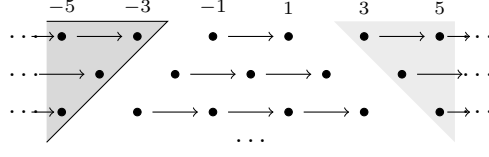
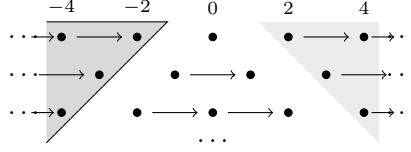
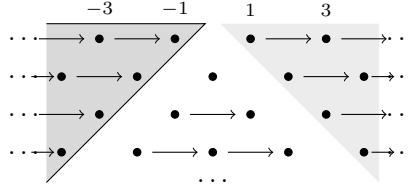
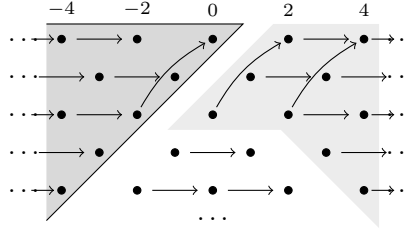
$$\begin{aligned} Q(n, \lambda_2) &\cong V(n, \lambda_2) \oplus V(n-1, \lambda_2) \oplus \dots \oplus V(1, \lambda_2) \oplus V(0, \lambda_2) \\ &\cong V(n, \lambda_2) \oplus Q(n-1, \lambda_2) \end{aligned}$$

In the same way, but using $Q(0, \lambda_2) \cong \mathbf{EA}(\Delta(-2, \lambda_2))$ in the first line (27) we can get that $Q(n, \lambda_2) \cong V(-n, \lambda_2) \oplus Q(n-1, \lambda_2)$. It follows that $V(n, \lambda_2) \cong V(-n, \lambda_2)$. $\square$

Similarly, one can prove the following analogue of Proposition 28:

Proposition 52. *Let $n, k \in \mathbb{Z}_{\geq 0}$ and $\lambda_2 \neq 0$. If $k \leq n$, then*

$$V(n, \lambda_2) \otimes L(k) = V(n-k, \lambda_2) \oplus V(n-k+2, \lambda_2) \oplus \dots \oplus V(n+k, \lambda_2).$$

FIGURE 5. $V(-1, \lambda_2) = \mathbf{EA}(\Delta(-3, \lambda_2))$ FIGURE 6. $V(0, \lambda_2) = \mathbf{EA}(\Delta(-2, \lambda_2))$ FIGURE 7. $V(0, \lambda_2) = \mathbf{EA}(\Delta(-1, \lambda_2))$ FIGURE 8. $V(1, \lambda_2) = \mathbf{EA}(\Delta(0, \lambda_2))$

If $k > n$, then

$$\begin{aligned} V(n, \lambda_2) \otimes L(k) &= V(0, \lambda_2) \oplus V(1, \lambda_2) \oplus \dots \oplus V(k-n-1, \lambda_2) \oplus \\ &\quad \oplus V(k-n, \lambda_2) \oplus V(k-n+2, \lambda_2) \oplus \dots \oplus V(k+n, \lambda_2). \end{aligned}$$

Since any simple $\mathfrak{g}$ -Harish-Chandra module is a quotient of some $Q(n, \lambda_2)$, we have proved:

Theorem 53. *Let V be a simple $\mathfrak{g}$ -Harish-Chandra module for $\mathfrak{s}$. Denote by λ_2 its central charge, and suppose $L(n)$, $n \in \mathbb{Z}_{\geq 0}$, is the minimal $\mathfrak{g}$ -type of V .*

- If $\lambda_2 \neq 0$, then $V \cong V(n, \lambda_2) \cong V(-n, \lambda_2)$.
- If $\lambda_2 = 0$, then $V \cong L(n)$ with the trivial $\mathfrak{h}$ -action.

In other words, $V(n, \lambda_2)$, $n \in \mathbb{Z}_{\geq 0}$, $\lambda_2 \in \mathbb{C} \setminus \{0\}$, together with the finite-dimensional simple $\mathfrak{g}$ -modules constitute a complete list of pairwise non-isomorphic simple $\mathfrak{g}$ -Harish-Chandra modules for $\mathfrak{s}$.

Remark 54. For the centerless Schrödinger Lie algebra $\bar{\mathfrak{s}}$, infinite-dimensional simple $\mathfrak{g}$ -Harish-Chandra modules do not exist. All simple $\mathfrak{g}$ -Harish-Chandra modules are given by $L(n)$, $n \in \mathbb{Z}_{\geq 0}$, with the trivial action of $\bar{\mathfrak{h}} := \mathfrak{h}/\mathbb{C}z$.

This follows from observing that the endomorphism ring of $\text{Ind}_{\bar{\mathfrak{g}}}^{\bar{\mathfrak{s}}}(L(0))$ is only $\mathbb{C}$ (similarly as in Proposition 39), and so all the universal modules have $\bar{\mathfrak{s}}$ -filtrations by simple finite-dimensional modules (similarly as in Theorem 42 for $\chi = 0$).

5.6. Extensions. Self-extensions of infinite-dimensional simple $\mathfrak{g}$ -Harish-Chandra modules (i.e. the ones having non-zero central charge) can be calculated in the same way as for the Takiff $\mathfrak{sl}_2$ case (Subsection 4.6):

Theorem 55. *Let V be a simple infinite-dimensional $\mathfrak{g}$ -Harish-Chandra module for $\mathfrak{s}$. Then $\text{Ext}^1(V, V) \cong \mathbb{C}$.*

5.7. Annihilators. We show that simple infinite-dimensional $\mathfrak{g}$ -Harish-Chandra modules again have the same annihilators as the corresponding Verma modules. The fact that annihilators of Verma modules for $\mathfrak{s}$ are centrally generated is already known, see [DLMZ14, Theorem 21].

Proposition 56. *Suppose $n \in \mathbb{Z}_{\geq 0}$ and $\lambda_2 \neq 0$.*

- (a) *The element p acts injectively on $V(n, \lambda_2)$.*
- (b) *The module $S_p \otimes_{U(\mathfrak{s})} V(n, \lambda_2)$ is isomorphic to the direct sum of Verma modules $\Delta(n-1, \lambda_2) \oplus \Delta(-n-2, \lambda_2)$.*

Proof. The proof is analogous to the proof of Proposition 33, but easier. So we will omit it. $\square$

From (the Schrödinger version of) Lemma 32, Proposition 56, the definition of $V(n, \lambda_2)$, and [DLMZ14, Theorem 21] we have:

Corollary 57. *Suppose $n \in \mathbb{Z}_{\geq 0}$ and $\lambda_2 \neq 0$. Then*

$$\text{Ann}(V(n, \lambda_2)) = \text{Ann}(\Delta(n-1, \lambda_2)) = \text{Ann}(\Delta(-n-2, \lambda_2)),$$

and these annihilators are centrally generated.

5.8. The action of finite-dimensional $\mathfrak{sl}_2$ -modules. Denote by $\mathcal{F}$ the monoidal category of finite-dimensional $\mathfrak{sl}_2$ -modules. For a fixed non-zero $\chi \in \mathbb{C}$, denote by $\mathcal{K}_{\chi}$ the category of semi-simple $\mathfrak{g}$ -Harish-Chandra $\mathfrak{s}$ -modules of central charge χ .

Proposition 58. *For each non-zero χ , the category $\mathcal{K}_{\chi}$ is a simple module category over $\mathcal{F}$.*

Proof. The fact that $\mathcal{K}_{\chi}$ is a module category over $\mathcal{F}$ follows directly from Proposition 52. Since $\mathcal{K}_{\chi}$ is semi-simple by definition, to show that it is a simple module category over $\mathcal{F}$ it is enough to show that, starting from any simple object of $\mathcal{K}_{\chi}$ and tensoring it with finite-dimensional $\mathfrak{sl}_2$ -modules, we can obtain any other simple object of $\mathcal{K}_{\chi}$ as a direct summand, up to isomorphism. This claim follows by combining Proposition 52 with Theorem 53. $\square$

We note that, by Proposition 52, the combinatorics of the $\mathcal{F}$ -module category $\mathcal{K}_\chi$ does not depend on χ . Furthermore, by comparing Propositions 28 and 52, we see that the combinatorics of the $\mathcal{F}$ -module category $\mathcal{H}_\chi$ is different from the combinatorics of the $\mathcal{F}$ -module category $\mathcal{K}_\chi$.

6. SOME GENERAL RESULTS ON $\mathfrak{g}$ -HARISH-CHANDRA MODULES

6.1. A sufficient condition for existence of simple infinite-dimensional $\mathfrak{g}$ -Harish-Chandra modules. Recall our general setup from Section 2, where $\mathfrak{L} \cong \mathfrak{g} \ltimes \mathfrak{r}$ was arbitrary finite-dimensional Lie algebra. Assume that a triangular decomposition (1) is fixed. Denote by $\mathfrak{r}_0$ the zero-weight space of $\mathfrak{r}$. Obviously, we have $\tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathfrak{r}_0$.

To prove the main theorem in this section, we need to use another variant of Enright's and Arkhipov's functors. It will be the same as **EA** from before, but without taking the locally finite for the positive root vector. Fix a simple reflection s , and the corresponding $\mathfrak{sl}_2$ -triple $\{f, h, e\} \subseteq \mathfrak{g}$. For an $\mathfrak{L}$ -module M , set

$$\mathbf{C}_s(M) := S_f \otimes_{U(\mathfrak{L})} M,$$

where, as before, S_f denotes the localized algebra $U(\mathfrak{L})_{(f)}$ modulo $U(\mathfrak{L})$. This functor commutes with the forgetful functor that forgets the $\mathfrak{r}$ -action. Moreover, it preserves $\mathfrak{g}$ -central characters.

Remark 59. Note that if we would twist the action on $\mathbf{C}_s(M)$ by the inner automorphism of $\mathfrak{L}$ that corresponds to the simple reflection s , we would get exactly the twisting functor $T_s(M)$ from [Ark04, AS03]. We conclude that if M is from the category $\mathcal{O}$ for $\mathfrak{g}$, then $\mathbf{C}_s(M)$ is also from the category $\mathcal{O}$ for $\mathfrak{g}$, but for another choice of Borel subalgebra.

Theorem 60. *Suppose that $[\mathfrak{L}, \mathfrak{r}] \cap \mathfrak{r}_0 \neq 0$. Then there exists an infinite-dimensional simple $\mathfrak{g}$ -Harish-Chandra module for $\mathfrak{L}$.*

Proof. Fix an element $c \in [\mathfrak{L}, \mathfrak{r}] \cap \mathfrak{r}_0$. Then $c \in \text{Nrad}(\mathfrak{L})$, so c must annihilate any simple finite-dimensional $\mathfrak{L}$ -module.

The idea is to start with the simple quotient of a Verma module for $\mathfrak{L}$, which has finite multiplicities of its $\mathfrak{g}$ -submodules (but possibly infinite-dimensional), and then use the functors $\mathbf{C}_s$ to obtain an $\mathfrak{L}$ -module that should have a finite-dimensional $\mathfrak{g}$ -submodule. Then the image of the universal $\mathfrak{L}$ -module in the constructed module should have only finite-dimensional $\mathfrak{g}$ -submodules with finite multiplicities. The element c will insure infinite-dimensionality.

Fix an antidominant, regular and integral $\lambda \in \mathfrak{h}^*$ and extend it to $\tilde{\lambda} \in \tilde{\mathfrak{h}}^*$ such that $\tilde{\lambda}(c) \neq 0$. Consider the Verma module $\Delta(\tilde{\lambda})$ as in (2), and its simple quotient $\mathbf{L}(\tilde{\lambda})$.

By considering the $\mathfrak{h}$ -weight spaces of $\Delta(\tilde{\lambda})$ (Proposition 4), it follows that as a $\mathfrak{g}$ -module, $\mathbf{L}(\tilde{\lambda})$ has a $\mathfrak{g}$ -direct summand $\Delta^{\mathfrak{g}}(\lambda)$ generated by the highest weight vector $\mathbf{v} \in \mathbf{L}(\tilde{\lambda})$, and this is the only $\mathfrak{g}$ -composition factor of $\mathbf{L}(\tilde{\lambda})$ of this $\mathfrak{g}$ -central character.

From the fact that $\Delta(\tilde{\lambda})$ and $\mathbf{L}(\tilde{\lambda})$ have finite-dimensional weight spaces, it follows that, when considered as $\mathfrak{g}$ -modules, they contain any simple $\mathfrak{g}$ -composition factor with at most finite multiplicity. Moreover, from this it follows that their components in any fixed $\mathfrak{g}$ -central character lie in the category $\mathcal{O}$ for $\mathfrak{g}$.

The element c acts on $\mathbf{v} \in \Delta^{\mathfrak{g}}(\lambda) \subseteq \mathbf{L}(\tilde{\lambda})$ by the scalar $\tilde{\lambda}(c) \neq 0$.

Choose a reduced expression $w_0 = s_1 s_2 \dots s_k$ of the longest element in the Weyl group for $\mathfrak{g}$, and set $C_{w_0} = C_{s_1} \circ C_{s_2} \circ \dots \circ C_{s_k}$. From Remark 59 and [AS03, Theorem 2.3] it follows that $C_{w_0}(\Delta^{\mathfrak{g}}(\lambda))$ is isomorphic to the dual dominant $\mathfrak{g}$ -Verma module for the opposite Borel subalgebra. From this, we can conclude $C_{w_0}(\Delta^{\mathfrak{g}}(\lambda))$ contains the finite-dimensional $\mathfrak{g}$ -submodule $L(\mu)$, where $\mu := w_0 \cdot \lambda$ is dominant, integral and regular. Moreover, by observing what is happening on the $\mathfrak{sl}_2$ -subalgebras of $\mathfrak{g}$, one can conclude that the lowest weight vector in $L(\mu)$ is given by

$$(28) \quad f_1^{-1} f_2^{-1} \dots f_k^{-1} \mathbf{v},$$

where f_i is the negative root vector corresponding to s_i .

It follows that $M := C_{w_0}(\mathbf{L}(\tilde{\lambda}))$ as a $\mathfrak{g}$ -module has also $L(\mu)$ as a $\mathfrak{g}$ -direct summand. Moreover, from Remark 59 it follows that $L(\mu)$ appears in M precisely once, and that each simple $\mathfrak{g}$ -module appears with at most finite multiplicity.

Consider $Q(\mu) := \text{Ind}_{\mathfrak{g}}^{\mathfrak{L}}(L(\mu)) = U(\mathfrak{L}) \otimes_{U(\mathfrak{g})} L(\mu)$. It has only finite-dimensional $\mathfrak{g}$ -composition factors, but possibly with infinite multiplicities.

By the universal property of the induction functor, we get a non-zero $\mathfrak{L}$ -homomorphism $\varphi: Q(\mu) \rightarrow M$, hitting the $\mathfrak{g}$ -submodule $L(\mu)$ in M . Denote by N the image of this map. It follows from the construction N is a $\mathfrak{g}$ -Harish-Chandra module, generated by its unique occurrence of the $\mathfrak{g}$ -type $L(\mu)$.

Furthermore, N has a unique simple quotient V , which contains this $\mathfrak{g}$ -type $L(\mu)$. Clearly, V is a simple $\mathfrak{g}$ -Harish-Chandra $\mathfrak{L}$ -module.

But also, V is infinite-dimensional. To see this, it is enough to check that c does not annihilate the vector (28). Observe that

$$c \cdot f_1^{-1} f_2^{-1} \dots f_k^{-1} \mathbf{v} = \tilde{\lambda}(c) \cdot f_1^{-1} f_2^{-1} \dots f_k^{-1} \mathbf{v} + \sum_{i=1}^k f_1^{-1} \dots [c, f_i^{-1}] \dots f_k^{-1} \mathbf{v}.$$

Since $[c, f_i^{-1}] = f_i^{-1} [f_i, c] f_i^{-1}$, each summand with $[c, f_i^{-1}] \neq 0$ will contain a factor from $\text{Nrad}(\mathfrak{L})$. Therefore, these terms cannot cancel with $\tilde{\lambda}(c) \cdot f_1^{-1} f_2^{-1} \dots f_k^{-1} \mathbf{v}$, and so the total result is non-zero. $\square$

We believe that the connection to the highest weight theory could be established in a more general setup, at least for the Takiff Lie algebras. Using the notation from Section 3, we formulate:

Conjecture 61. *Let $\mathfrak{T}$ be a Takiff Lie algebra attached to a semi-simple Lie algebra $\mathfrak{g}$. For a “generic” $\chi: \overline{Z(\mathfrak{g})} \rightarrow \mathbb{C}$, there is $\lambda \in \bar{\mathfrak{h}}^*$ such that $\mathbf{EA}_{w_0}(\Delta(-2\rho, \lambda)) \cong Q(0, \chi)$.*

Here ρ is the half-sum of all elements in $\Delta^+(\mathfrak{g}, \mathfrak{h})$, and $\mathbf{EA}_{w_0}$ should be defined as the composition $\mathbf{EA}_{s_1} \circ \mathbf{EA}_{s_2} \circ \dots \circ \mathbf{EA}_{s_k}$, where $w_0 = s_1 s_2 \dots s_k$ is a fixed a reduced expression of the longest element in the Weyl group for $\mathfrak{g}$. Each $\mathbf{EA}_s$ should be defined as in Definition 20, i.e., as $\mathbf{EA}_s := {}^e(C_s(-))$ for the $\mathfrak{sl}_2$ -triple $\{f, h, e\}$ corresponding to s . It is not a priori clear that such $\mathbf{EA}_{w_0}$ does not depend on the choice of a reduced expression.

Conjecture 61 was already proved for the Takiff $\mathfrak{sl}_2$ case in Section 4. Analogous statement is proved also for the Schrödinger Lie algebra in Section 5.

6.2. On classification of simple $\mathfrak{g}$ -Harish-Chandra modules for generalized Takiff Lie algebras. In this subsection we consider a finite-dimensional Lie algebra $\mathfrak{L}$ with a fixed Levi decomposition $\mathfrak{L} \cong \mathfrak{g} \ltimes \mathfrak{r}$ and assume that:

$\mathfrak{r}$ is abelian.

It is reasonable to call such algebras *generalized Takiff Lie algebras*. In analogy to Subsection 3.2, we consider the *purely radical* part $\overline{Z(\mathfrak{L})} := Z(\mathfrak{L}) \cap U(\mathfrak{r})$ of the center $Z(\mathfrak{L})$ of $U(\mathfrak{L})$. Since $\mathfrak{r}$ is assumed to be abelian, it is obvious that $\overline{Z(\mathfrak{L})} = U(\mathfrak{r})^{\mathfrak{g}}$, the $\mathfrak{g}$ -invariants in $U(\mathfrak{r}) \cong \text{Sym}(\mathfrak{r})$ with respect to the adjoint action. For brevity, algebra homomorphisms $\chi : \overline{Z(\mathfrak{L})} \rightarrow \mathbb{C}$ will be loosely called *radical central characters*.

For $\lambda \in \mathfrak{h}^*$ integral, dominant and regular, we have the *universal module*

$$Q(\lambda) := \text{Ind}_{\mathfrak{g}}^{\mathfrak{L}} L(\lambda) = U(\mathfrak{L}) \otimes_{U(\mathfrak{g})} L(\lambda) \cong U(\mathfrak{r}) \otimes_{\mathbb{C}} L(\lambda) \cong \text{Sym}(\mathfrak{r}) \otimes_{\mathbb{C}} L(\lambda).$$

Clearly, it is $\mathfrak{g}$ -locally finite. Completely analogously to Proposition 6, one can show that $\text{End } Q(0) \cong \overline{Z(\mathfrak{L})}$. Given also a radical central character $\chi : \overline{Z(\mathfrak{L})} \rightarrow \mathbb{C}$, we define

$$Q(\lambda, \chi) := Q(\lambda) / \mathfrak{m}_{\chi} Q(\lambda),$$

where $\mathfrak{m}_{\chi} := \text{Ker } \chi$ is the maximal ideal in $\overline{Z(\mathfrak{L})}$ corresponding to χ .

From [Kna88, Proposition 6.5] we have $Q(\lambda) \cong Q(0) \otimes L(\lambda)$, and from this and right-exactness of tensor product we conclude $Q(\lambda, \chi) \cong Q(0, \chi) \otimes L(\lambda)$. In these formulas $L(\lambda)$ is considered to be an $\mathfrak{L}$ -module with the trivial action of $\mathfrak{r}$.

By construction, the action of $\mathfrak{g}$ on $Q(0, \chi)$ is locally finite, with the trivial module $L(0)$ having multiplicity exactly 1 in $Q(0, \chi)$. Since $Q(0, \chi)$ is generated by this unique copy of $L(0)$, it follows that $Q(0, \chi)$ has a unique simple quotient, which we denote by $V(0, \chi)$.

From Theorem 11 we know that $Q(0, \chi) = V(0, \chi)$ when $\mathfrak{L}$ is the Takiff $\mathfrak{sl}_2$ and χ is not the radical central character of the trivial module, but generally this fails, for example when $\mathfrak{r} \cong L(4)$ as a $\mathfrak{g}$ -module, as we will see in Subsection 7.3.

We will use the notion of Gelfand-Kirillov dimension, for the definition and basic properties see [CCM, 2.3] or [KL00]. The Gelfand-Kirillov dimension of a finitely generated module M will be denoted by $\text{GK}(M)$ and the Bernstein number by $e(M)$.

Proposition 62. *Fix a radical central character χ .*

- *$V(0, \chi)$ is the unique simple $\mathfrak{L}$ -module having both radical central character χ and the trivial $\mathfrak{g}$ -module as one of $\mathfrak{g}$ -types.*
- *If $[\mathfrak{g}, \mathfrak{r}] = \mathfrak{r}$, then $V(0, \chi)$ is either the trivial $\mathfrak{L}$ -module (precisely when χ is the radical central character of the trivial $\mathfrak{L}$ -module), or infinite-dimensional.*
- *$V(0, \chi)$ is a $\mathfrak{g}$ -Harish-Chandra module.*

Proof. The first claim follows from universal property of the induction functor.

For the second claim, recall that $[\mathfrak{g}, \mathfrak{r}] = \text{Nrad } \mathfrak{L}$ is the intersection of all kernels of simple finite-dimensional $\mathfrak{L}$ -modules (see 2). Therefore any simple finite-dimensional $\mathfrak{L}$ -module necessarily has the same radical central character as the trivial module $\mathfrak{L}$ -module. From the uniqueness in the first part it follows that $V(0, \chi)$ is either trivial or infinite-dimensional.

It remains to prove that $V(0, \chi)$ is a $\mathfrak{g}$ -Harish-Chandra module. It is $\mathfrak{g}$ -locally finite by construction. Assume that $[V(0, \chi) : L] = \infty$ for some simple finite-dimensional

$\mathfrak{g}$ -module L . The module $V(0, \chi) \otimes L^*$ is also $\mathfrak{g}$ -locally finite and finitely generated. By [Jan83, Lemma 8.8] we have

$$(29) \quad \text{GK}(V(0, \chi) \otimes L^*) = \text{GK}(V(0, \chi)) \quad \text{and} \quad e(V(0, \chi) \otimes L^*) = \dim L \cdot e(V(0, \chi)).$$

Moreover,

$$\begin{aligned} [V(0, \chi) \otimes L^* : L(0)] &= \text{Hom}_{\mathfrak{g}}(L(0), V(0, \chi) \otimes L^*) = \\ &= \text{Hom}_{\mathfrak{g}}(L(0) \otimes L, V(0, \chi)) = [V(0, \chi) : L] = \infty. \end{aligned}$$

By uniqueness in the first part, it follows that $V(0, \chi)$ appears in $V(0, \chi) \otimes L^*$ infinitely many times. This is a contradiction with (29) and [CCM, Lemma 8]. $\square$

Remark 63. One can check that Gelfand-Kirillov dimensions of all infinite-dimensional simple $\mathfrak{g}$ -Harish-Chandra modules for the Takiff $\mathfrak{sl}_2$ (Section 4) as well for the Schrödinger Lie algebra (Section 5) are 2.

Proposition 64. *Let χ be a radical central character and L a simple finite-dimensional $\mathfrak{g}$ -module. Then $\mathfrak{L}$ has a simple $\mathfrak{g}$ -Harish-Chandra module with the radical central character χ and having L as one of $\mathfrak{g}$ -types.*

Moreover, the number of (isomorphism classes of) simple $\mathfrak{g}$ -Harish-Chandra modules with the radical central character χ and having L as one of its $\mathfrak{g}$ -types is at most

$$(30) \quad \sum_M l_{M,L}^L [V(0, \chi) : M] < \infty,$$

where M runs through the set of isomorphism classes of simple finite-dimensional $\mathfrak{g}$ -modules and $l_{M,L}^L$ is the Littlewood-Richardson coefficient, i.e. the multiplicity of L in $M \otimes L$.

We note that only finitely many summands in (30) are non-zero, as $l_{M,V}^V \neq 0$ if and only if M appears as a summand of $V \otimes V^*$.

Proof. Since we know from Proposition 62 that $V(0, \chi)$ is a $\mathfrak{g}$ -Harish-Chandra module having $L(0)$ as a $\mathfrak{g}$ -type, it follows that $L \otimes V(0, \chi)$ is also a $\mathfrak{g}$ -Harish-Chandra module, having L as one of $\mathfrak{g}$ -types. This implies the first part of the proposition.

Conversely, let N be a simple $\mathfrak{g}$ -Harish-Chandra module with the radical central character χ and having L as one of $\mathfrak{g}$ -types. Then $L^* \otimes N$ is a $\mathfrak{g}$ -Harish-Chandra module with the radical central character χ and having $L(0)$ as one of $\mathfrak{g}$ -types, and therefore contains $V(0, \chi)$ as a subquotient.

The above arguments, combined with the biadjunction $(L \otimes -, L^* \otimes -)$, imply that any simple $\mathfrak{g}$ -Harish-Chandra module with the radical central character χ and having L as one of $\mathfrak{g}$ -types is a subquotient of $L \otimes V(0, \chi)$. The latter is a $\mathfrak{g}$ -Harish-Chandra module in which the multiplicity of L is bounded by the expression in (30) (see the proof of Proposition 7). This implies the second part of the proposition. $\square$

7. ON $\mathfrak{sl}_2$ -HARISH-CHANDRA MODULES FOR OTHER CONFORMAL GALILEI ALGEBRAS

7.1. Conformal Galilei algebras. By a *conformal Galilei algebra* we mean a semi-direct product $\mathfrak{L}^n := \mathfrak{sl}_2 \ltimes L(n)$, where $n \in \mathbb{Z}_{\geq 0}$. Here $L(n)$ is an abelian ideal on which $\mathfrak{g} := \mathfrak{sl}_2$ acts in the obvious way. For a more general definition and various central extensions, see [LMZ14, AIM19, GK12].

Note that we have $\mathfrak{L}^0 \cong \mathfrak{gl}_2$, $\mathfrak{L}^1$ is the centerless Schrödinger Lie algebra, and $\mathfrak{L}^2$ is the Takiff $\mathfrak{sl}_2$.

We have $\text{Nrad}(\mathfrak{L}^n) = \mathfrak{r} = L(n)$, and so $[\mathfrak{L}^n, \mathfrak{r}] \cap \mathfrak{r}_0 \cong \mathbb{C}$ if and only if n is even (i.e., $L(n)$ is odd-dimensional), otherwise $[\mathfrak{L}^n, \mathfrak{r}] \cap \mathfrak{r}_0 = 0$.

Denote by $v_n, v_{n-2}, \dots, v_{-n}$ a basis of $L(n)$ such that each v_i is a weight vector of weight i , and $[e, v_{n-2i}] = (n - i + 1)v_{n-2i+2}$, for $i \in \{1, 2, \dots, n\}$.

7.2. The Lie algebra $\mathfrak{L}^3 = \mathfrak{sl}_2 \ltimes L(3)$. Since $\mathfrak{r} = L(3)$ has trivial zero-weight space, the assumption in Theorem 60 is not satisfied. Nevertheless, we will show that simple infinite-dimensional $\mathfrak{g}$ -Harish-Chandra modules exist. So the converse of Theorem 60 is not true. This suggest that the highest weight theory is not enough to obtain and classify $\mathfrak{g}$ -Harish-Chandra modules for any Lie algebra.

From the classical invariant theory it is well known that $\text{Sym}(\mathfrak{r})^{\mathfrak{g}}$ is generated by only one element C , homogeneous of degree 4, the so called cubic discriminant (see e.g. [Hil93, Lecture XVII]):

$$C = v_{-1}^2 v_1^2 - 27v_{-3}^2 v_3^2 - 4v_{-1}^3 v_3 - 4v_{-1} v_3^3 + 18v_{-3} v_{-1} v_1 v_3.$$

This expression of C is just for the record, we will not use it in the computations. We identify radical central characters with their value on C .

For $\chi \in \mathbb{C}$ we have the universal $\mathfrak{L}^3$ -module $Q(0, \chi) = Q(0)/(C - \chi)$.

Proposition 65. *For $\chi \neq 0$, the module $Q(0, \chi)$ is $\mathfrak{g}$ -Harish-Chandra. The multiplicities of its $\mathfrak{g}$ -types are given by $[Q(0, \chi): L(k)] = k - 2\lfloor \frac{k+2}{3} \rfloor + 1$.*

Proof. From the main result of [FO05] it follows easily that $\text{Sym}(\mathfrak{r})$ is free as a module over $\text{Sym}(\mathfrak{r})^{\mathfrak{g}} = \mathbb{C}[C]$. This implies that the $\mathfrak{g}$ -structure of $Q(0, \chi)$ does not depend on the choice of χ . So in particular, it is enough to prove the finite-multiplicity statement for $Q(0, 0)$, which is as a $\mathfrak{g}$ -module isomorphic to $\text{Sym}(\mathfrak{r})/C \cdot \text{Sym}(\mathfrak{r})$.

But these are now graded modules, so we can subtract their graded characters. More precisely, for $d \geq 4$ and $k \geq 0$ we have

$$(31) \quad [Q(0, 0)^d: L(k)] = [\text{Sym}^d(\mathfrak{r}): L(k)] - [\text{Sym}^{d-4}(\mathfrak{r}): L(k)],$$

where $(-)^d$ denotes the homogeneous part of degree d .

Using [HHLS18], one can calculate the right-hand side of (31). For non-negative integers a, b, c let $p(a, b, c)$ denote the number of partitions of c into at most b parts, and each part bounded above by a . Denote $N(a, b, c) := p(a, b, c) - p(a, b, c - 1)$ if $c \geq 1$, and set $N(a, b, 0) := 1$. By [HHLS18, Theorem 3.1], the multiplicity (31) is equal to 0 if $k \notin \{3d, 3d - 2, 3d - 4, \dots\}$, and to

$$N\left(d, 3, \frac{3d - k}{2}\right) - N\left(d - 4, 3, \frac{3d - k}{2} - 6\right)$$

otherwise. But the latter is also equal to 0 whenever $k < d - 4$, by using the formulas in [HHLS18, Corollary 3.2]. It follows that any $L(k)$ can appear in $Q(0, 0)$ in at most degree $k + 4$, hence only finitely many times.

Using the same formulas in [HHLS18, Corollary 3.2], one can derive the multiplicity formula from the statement of the proposition. We omit the details. $\square$

Either from Proposition 62, or from Proposition 65, we have:

Corollary 66. *The unique simple quotient $V(0, \chi)$ of $Q(0, \chi)$ is $\mathfrak{g}$ -Harish-Chandra. It is infinite-dimensional if $\chi \neq 0$.*

We do not know whether $Q(0, \chi) = V(0, \chi)$, i.e., whether $Q(0, \chi)$ is already simple (for $\chi \neq 0$), as was in the Takiff $\mathfrak{sl}_2$ and the Schrödinger cases.

7.3. The Lie algebra $\mathfrak{L}^4 = \mathfrak{sl}_2 \ltimes L(4)$. Consider now the algebra $\mathfrak{L}^4$. In this subsection we classify simple $\mathfrak{g}$ -Harish-Chandra modules for $\mathfrak{L}^4$ which appear in Enright-Arkhipov completions of simple highest weight modules.

The algebra $\text{Sym}(\mathfrak{r})^{\mathfrak{g}}$ is generated by two algebraically independent elements, homogeneous of degrees 2 and 3 (see e.g. [Hil93, Lecture XVIII]):

$$\begin{aligned} C_2 &= v_0^2 - 3v_{-2}v_2 + 12v_{-4}v_4, \\ C_3 &= v_0^3 - \frac{9}{2}v_{-2}v_0v_2 + \frac{27}{2}v_{-2}^2v_4 + \frac{27}{2}v_{-4}v_2^2 - 36v_{-4}v_0v_4. \end{aligned}$$

Recall also, from [LMZ14, Theorem 4], the structure of simple highest weight $\mathfrak{L}^4$ -modules. Let $\tilde{\mathfrak{h}}$ denote the (generalized) Cartan subalgebra of $\mathfrak{L}^4$ spanned by h and v_0 . If $\lambda \in \tilde{\mathfrak{h}}^*$ is such that $\lambda(v_0) = 0$, then $\mathfrak{r}$ annihilates the corresponding simple highest weight module $\mathbf{L}(\lambda)$. If $\lambda \in \tilde{\mathfrak{h}}^*$ is such that $\lambda(v_0) \neq 0$, then the restriction of $\mathbf{L}(\lambda)$ to $\mathfrak{g}$ has a multiplicity free Verma filtration with subquotients of the form $\Delta^{\mathfrak{g}}(\lambda - n\alpha)$, where $n \in \mathbb{Z}_{\geq 0}$ and α is the root corresponding to $e \in \mathfrak{g}$. Note that the elements C_2 and C_3 act on $\mathbf{L}(\lambda)$ as the scalars $\lambda(v_0)^2$ and $\lambda(v_0)^3$, respectively.

Denote by $\mathcal{F}$ the semi-simple additive category generated by simple subquotients of Enright-Arkhipov completions of simple highest weight $\mathfrak{L}^4$ -modules. Note that all modules in $\mathcal{F}$ are $\mathfrak{g}$ -Harish-Chandra modules for $\mathfrak{L}^4$. Our main result of this subsection is the following theorem.

- Theorem 67.** (a) For each $\tilde{\lambda} \in \mathbb{C} \setminus \{0\}$ and for each $i \in \mathbb{Z}_{>0}$, there is a unique, up to isomorphism, simple object $V(i, \tilde{\lambda})$ in $\mathcal{F}$ on which C_j , where $j = 2, 3$, acts via $\tilde{\lambda}^j$ and which has $\mathfrak{g}$ -types $L(i), L(i+2), L(i+4), \dots$, all multiplicity free.
- (b) For each $\tilde{\lambda} \in \mathbb{C} \setminus \{0\}$, there is a unique, up to isomorphism, simple object $V'(0, \tilde{\lambda})$ in $\mathcal{F}$ on which C_j , where $j = 2, 3$, acts via $\tilde{\lambda}^j$ and which has $\mathfrak{g}$ -types $L(0), L(4), L(8), \dots$, all multiplicity free.
- (c) For each $\tilde{\lambda} \in \mathbb{C} \setminus \{0\}$, there is a unique, up to isomorphism, simple object $V'(2, \tilde{\lambda})$ in $\mathcal{F}$ on which C_j , where $j = 2, 3$, acts via $\tilde{\lambda}^j$ and which has $\mathfrak{g}$ -types $L(2), L(6), L(10), \dots$, all multiplicity free.
- (d) The modules above provide a complete and irredundant list of representatives of isomorphism classes of simple objects in $\mathcal{F}$.
- (e) Let V be a simple $\mathfrak{g}$ -Harish-Chandra module on which C_j , where $j = 2, 3$, acts via $\tilde{\lambda}^j$, for some $\tilde{\lambda} \in \mathbb{C} \setminus \{0\}$. Then V belongs to $\mathcal{F}$.

To prove this result, we will need some preparation. The following lemma extends [HHLS18, Corollary 3.5] (note that the case treated in the lemma below is referred to as “complicated” in [HHLS18]).

Lemma 68. For every non-negative integer k , we have

$$\sum_{s=0}^{\lfloor \frac{2k-1}{4} \rfloor} \left(\left\lfloor \frac{2k-4s-1}{2} \right\rfloor - \left\lfloor \frac{2k-4s-2}{3} \right\rfloor \right) - \sum_{s=0}^{k-2} \left(\left\lfloor \frac{s}{2} \right\rfloor - \left\lfloor \frac{s-1}{3} \right\rfloor \right) = 0.$$

Proof. Using computer, it is easy to check that the claim of the lemma is true for small values of k (we checked this independently and by different methods using Scilab and SageMath up to $k = 200$). After that, one can do induction on k with induction step 12. So, we write $k = 12a + r$ and consider each r separately. Let $S(k)$ denote the left

hand side of the formula. For $k > 12$, the value $S(k) - S(k - 12)$ can be written as a polynomial in a (the polynomial itself depends on r) of degree at most two.

From the original computation it follows that $S(k) - S(k - 12)$ vanishes for enough values of a to conclude that $S(k) - S(k - 12)$ is identically 0. The claim follows. $\square$

Remark 69. The results of [HHLS18] say that Lemma 68 is equivalent to the fact that, for each $k \geq 0$, the set Λ_1 of all vectors (a, b, c, d, e) with non-negative integer coefficients satisfying $a + b + c + d + e = k$ and $2a + b - d - 2e = 1$ has the same cardinality as the set Λ_2 of all vectors (a, b, c, d, e) with non-negative integer coefficients satisfying $a + b + c + d + e = k$ and $2a + b - d - 2e = 2$. We give here explicitly a bijection between these sets. First, note that $\{(a, b, c, d, e) \in \Lambda_1 : e \neq 0\}$ maps bijectively to $\{(a', b', c', d', e') \in \Lambda_2 : d' \neq 0\}$ by

$$(a, b, c, d, e) \mapsto (a, b, c, d + 1, e - 1).$$

The remainder $\{(a, b, c, d, 0) \in \Lambda_1\}$ maps bijectively to $\{(a', b', c', 0, e') \in \Lambda_2\}$ by the formula

$$(a, b, c, d, 0) \mapsto \left(b, 2a, c, 0, \frac{b + d - 1}{2}\right).$$

Lemma 70. *Let V be a simple infinite-dimensional $\mathfrak{g}$ -Harish-Chandra module for $\mathfrak{L}^4$. Then each v_i acts injectively on V .*

Proof. Since the adjoint action of v_i on $\mathfrak{L}^4$ is locally nilpotent, the action of v_i on each simple $\mathfrak{L}^4$ -module is either injective or locally nilpotent, cf. [DMP, Section 3]. Note that the action of both e and f on V is locally nilpotent by definition. Let $x \in V$ be such that $v_i \cdot x = 0$, for some $i \neq 4$, and $e^m \cdot v = 0$. Then $\text{ad}^m(v_i)(e^m) \cdot x = 0$ and it is easy to check that this implies that $v_{i+2}^m \cdot x = 0$. That is, the action of v_{i+2} is locally nilpotent. Applying similar arguments using e and f , we get that the action of all v_j 's is locally nilpotent.

As all v_j 's commute, V must contain some non-zero x which is killed by all the v_j 's. Since the adjoint action of e preserves the set of the v_j 's, we can even assume that e kills x . But then this means that V is a highest weight module. Being also a $\mathfrak{g}$ -Harish-Chandra module, this implies that V must be finite-dimensional, a contradiction. $\square$

For $\chi_2, \chi_3 \in \mathbb{C}$, we recall the universal $\mathfrak{L}^4$ -module

$$Q(0, \chi_2, \chi_3) = Q(0) / (C_2 - \chi_2, C_3 - \chi_3),$$

and its unique simple quotient $V(0, \chi_2, \chi_3)$ containing $L(0)$.

Proposition 71. *Fix $\tilde{\lambda} \in \mathbb{C} \setminus \{0\}$. The module $V(0, \tilde{\lambda}^2, \tilde{\lambda}^3)$ is a simple infinite-dimensional $\mathfrak{g}$ -Harish-Chandra module. Its $\mathfrak{g}$ -types are $L(0), L(4), L(8) \dots$, each occurring with multiplicity one.*

Proof. By [HHLS18, Theorem 3.1 and Corollary 3.4], the multiplicity of $L(2)$ in $Q(0)$ is given by the left hand side of the formula from Lemma 68. Therefore, by Lemma 68, $L(2)$ does not appear in $Q(0)$.

Now, take λ such that $\lambda(h) = -2$ and $\lambda(v_0) = \tilde{\lambda}$. Then $\mathbf{EA}(\mathbf{L}(\lambda))$ is a $\mathfrak{g}$ -Harish-Chandra module having multiplicity-free $\mathfrak{g}$ -types $L(0), L(2), L(4), \dots$. As v_4 commutes with e , from Lemma 70 it follows that v_4 sends each non-zero highest weight vector of $L(i)$ inside $\mathbf{EA}(\mathbf{L}(\lambda))$ to a non-zero highest weight vector of $L(i+4)$ inside $\mathbf{EA}(\mathbf{L}(\lambda))$. Note that all simple subquotients of $\mathbf{EA}(\mathbf{L}(\lambda))$ must be infinite-dimensional as the central characters of $\mathbf{EA}(\mathbf{L}(\lambda))$ is different, by construction, from the central characters of simple finite-dimensional $\mathfrak{L}^4$ -modules.

By the universal property of $Q(0)$, the inclusion of $L(0)$ in $\mathbf{EA}(\mathbf{L}(\lambda))$ gives rise to a non-zero homomorphism from $Q(0)$ to $\mathbf{EA}(\mathbf{L}(\lambda))$. The image V of this homomorphism does not contain $L(2)$, as was established in the first paragraph of the proof. Therefore, from Lemma 70 it follows that V has $\mathfrak{g}$ -types $L(0), L(4), L(8), \dots$ and the quotient $\mathbf{EA}(\mathbf{L}(\lambda))/V$ has $\mathfrak{g}$ -types $L(2), L(6), L(10), \dots$. In fact, from Lemma 70 and the above remark that all simple subquotients of $\mathbf{EA}(\mathbf{L}(\lambda))$ must be infinite-dimensional, it follows that both V and $\mathbf{EA}(\mathbf{L}(\lambda))/V$ are simple modules.

This implies that $V \cong V(0, \tilde{\lambda}^2, \tilde{\lambda}^3)$ and the claim of the lemma follows. $\square$

Lemma 72. *If $\lambda(v_0) \neq 0$, then, in the category of h -weight $\mathfrak{L}^4$ -modules, we have the vanishing $\text{Ext}^1(\mathbf{L}(\lambda - \alpha), \mathbf{L}(\lambda)) = 0$.*

Proof. Let $\mathbf{L}(\lambda) \hookrightarrow M \twoheadrightarrow \mathbf{L}(\lambda - \alpha)$ be a short exact sequence in the category of h -weight $\mathfrak{L}^4$ -modules. Consider the vector space $X := M_\lambda \oplus M_{\lambda - \alpha}$ and note that it is killed by v_4 . Therefore this vector space is a module over the polynomial algebra A in e and v_2 . The space of first self-extensions for each simple A -module is two-dimensional. Since $\mathbf{L}(\lambda)$ is simple, the submodule $Y := \mathbf{L}(\lambda)_\lambda \oplus \mathbf{L}(\lambda)_{\lambda - \alpha}$ of X is indecomposable. Since $\mathbf{L}(\lambda)_{\lambda - \alpha}$ has dimension 2, Y is the universal self-extension of the trivial A -module. This implies that, in the category of A -modules, the first extension from $\mathbf{L}(\lambda - \alpha)_{\lambda - \alpha}$ to Y coming from the socle of Y vanishes. Consequently, M must have a non-zero vector of weight $\lambda - \alpha$ which is killed by both e and v_2 . As the adjoint action of v_0 leaves the span of e and v_2 invariant, it follows that M contains a highest weight vector of weight $\lambda - \alpha$. Consequently, M splits, proving the claim. $\square$

Now we are ready to prove Theorem 67.

Proof of Theorem 67. We take λ such that $\lambda(h) = -2$ and $\lambda(v_0) = \tilde{\lambda}$.

The $\mathfrak{L}^4$ -module $V'(0, \tilde{\lambda}) := V(0, \tilde{\lambda}^2, \tilde{\lambda}^3)$ and the module $V'(2, \tilde{\lambda}) := \mathbf{EA}(L(\lambda))/V$, cf. the proof of Proposition 71, are already constructed. Note that the proof of Proposition 71 implies

$$\text{Ext}^1(V'(0, \tilde{\lambda}), V'(2, \tilde{\lambda})) = 0$$

in the category of $\mathfrak{g}$ -Harish-Chandra modules. As usual, on the category of $\mathfrak{g}$ -Harish-Chandra modules we have the restricted duality, which we denote by $\star$, that maps $\bigoplus_i L(i)^{\oplus m_i}$ to $\bigoplus_i (L(i)^*)^{\oplus m_i}$. The fact that $V'(0, \tilde{\lambda})$ is self-dual follows directly from its uniqueness given by construction. Applying $\star$, we obtain

$$\text{Ext}^1(V'(2, \tilde{\lambda})^\star, V'(0, \tilde{\lambda})) = 0,$$

where the modules $V'(2, \tilde{\lambda})$ and $V'(2, \tilde{\lambda})^\star$ have the same $\mathfrak{g}$ -types. If we assume that $V'(2, \tilde{\lambda}) \not\cong V'(2, \tilde{\lambda})^\star$, then, from Proposition 64, it follows that these are the only simple $\mathfrak{g}$ -Harish-Chandra modules having $L(2)$ as a $\mathfrak{g}$ -type.

Consider now the module $L(2) \otimes V'(0, \tilde{\lambda})$. By adjunction, this must have both $V'(2, \tilde{\lambda})$ and $V'(2, \tilde{\lambda})^\star$ as simple subquotients, both with multiplicity one. The remaining $\mathfrak{g}$ -types are $L(4), L(8), L(10), \dots$. If V is a subquotient of $L(2) \otimes V'(0, \tilde{\lambda})$ whose $\mathfrak{g}$ -types form a subset of these remaining $\mathfrak{g}$ -types, then V , by adjunction, cannot be in the top or socle of $L(2) \otimes V'(0, \tilde{\lambda})$ as $L(2) \otimes V$ does not have $L(0)$ as a simple subquotient. This implies that $L(2) \otimes V'(0, \tilde{\lambda})$ must have socle $V'(2, \tilde{\lambda})$ and top $V'(2, \tilde{\lambda})^\star$ or vice-versa. However, since both $V'(0, \tilde{\lambda})$ and $L(2)$ are self-dual, so is $L(2) \otimes V'(0, \tilde{\lambda})$, a contradiction. Therefore $V'(2, \tilde{\lambda}) \cong V'(2, \tilde{\lambda})^\star$ and thus

$$(32) \quad \mathbf{EA}(\mathbf{L}(\lambda)) \cong V'(0, \tilde{\lambda}) \oplus V'(2, \tilde{\lambda}).$$

Consider now the module $L(1) \otimes V'(0, \tilde{\lambda})$. It has $\mathfrak{g}$ -types $L(1), L(3), L(5), \dots$, all multiplicity-free. We claim that $L(1) \otimes V'(0, \tilde{\lambda})$ is a simple $\mathfrak{g}$ -module which we can declare to be $V(1, \tilde{\lambda})$. Assume that $L(1) \otimes V'(0, \tilde{\lambda})$ is not simple and let V be a submodule or a quotient of $L(1) \otimes V'(0, \tilde{\lambda})$ which does not have $L(1)$ as its $\mathfrak{g}$ -type. Using the self-adjointness of $L(1) \otimes -$, by adjunction we have a non-zero homomorphism between $V'(0, \tilde{\lambda})$ and $L(1) \otimes V$. However, the latter is not possible as $L(0)$ is not a $\mathfrak{g}$ -type of $L(1) \otimes V$ due to our definition of V . This shows that $L(1) \otimes V'(0, \tilde{\lambda})$ is simple.

From Lemma 72, it follows that $L(1) \otimes \mathbf{L}(\lambda) \cong \mathbf{L}(\lambda - \frac{1}{2}\alpha) \oplus \mathbf{L}(\lambda + \frac{1}{2}\alpha)$. As $\mathbf{EA}$ commutes with $L(1) \otimes -$, it follows that

$$L(1) \otimes \mathbf{EA}(\mathbf{L}(\lambda)) \cong \mathbf{EA}(\mathbf{L}(\lambda - \frac{1}{2}\alpha)) \oplus \mathbf{EA}(\mathbf{L}(\lambda + \frac{1}{2}\alpha)).$$

By comparing the $\mathfrak{g}$ -types, we see that

$$\mathbf{EA}(\mathbf{L}(\lambda - \frac{1}{2}\alpha)) \cong V(1, \tilde{\lambda}) \quad \text{or} \quad \mathbf{EA}(\mathbf{L}(\lambda + \frac{1}{2}\alpha)) \cong V(1, \tilde{\lambda}).$$

Consider the first case, the second one is similar. By adjunction, we have

$$\text{Hom}(V(1, \tilde{\lambda}), L(1) \otimes \mathbf{EA}(\mathbf{L}(\lambda))) \cong \text{Hom}(L(1) \otimes V(1, \tilde{\lambda}), \mathbf{EA}(\mathbf{L}(\lambda))).$$

By the above, $L(1) \otimes V(1, \tilde{\lambda})$ has $\mathbf{EA}(\mathbf{L}(\lambda))$ as a direct summand and the endomorphism of $\mathbf{EA}(\mathbf{L}(\lambda))$ has dimension two. Consequently, $\text{Hom}(V(1, \tilde{\lambda}), \mathbf{L}(\lambda + \frac{1}{2}\alpha))$ must be non-zero, which implies

$$\mathbf{EA}(\mathbf{L}(\lambda + \frac{1}{2}\alpha)) \cong V(1, \tilde{\lambda})$$

by comparing the $\mathfrak{g}$ -types of these modules.

Using, inductively, arguments similar to the ones used above, we construct modules

$$V(i, \tilde{\lambda}) \cong \mathbf{EA}(\mathbf{L}(\lambda + \frac{i}{2}\alpha))$$

for $i > 1$. The proof of Theorem 67 is now completed easily using construction and adjunction. $\square$

Remark 73 (A sketch of an alternative proof of the splitting (32)). Denote by $\mathfrak{b}$ the span of h and e , and consider the polynomials $\mathbb{C}[x]$ as $(\mathfrak{b} \ltimes L(4))$ -module by declaring: $h \cdot x^k = (-2 - 2k)x^k$, e acts as $\frac{\partial}{\partial x}$, v_4 and v_2 annihilate everything, v_0 multiplies by $3\tilde{\lambda}$, v_{-2} multiplies by $6\tilde{\lambda}x$, and v_{-2} multiplies by $3\tilde{\lambda}x^2$ (this is known as the Fock module). From [LMZ14, Theorem 4(ii)] it follows that $\mathbf{L}(\lambda) \cong \text{Ind}_{\mathfrak{b} \ltimes L(4)}^{\mathfrak{g}^4} \mathbb{C}[x]$ (recall that $\lambda(h) = -2$ and $\lambda(v_0) = \tilde{\lambda}$). From this, we have that $\mathbf{EA}(\mathbf{L}(\lambda))$ has $\mathfrak{g}$ -types $L(0), L(2), L(4), \dots$, and the lowest weight vector in each $L(2k)$ is $f^{-1} \otimes x^k$. By a long and tedious, but straightforward computation, one can see that

$$\begin{aligned} v_4 f^{-1} \otimes x^k &= \frac{3\tilde{\lambda}}{4(k+1)(k+2)(2k+1)(2k+3)} \cdot e^4 f^{-1} \otimes x^{k+2} \\ &\quad - \frac{3\tilde{\lambda}}{2(2k+3)(2k-1)} \cdot e^2 f^{-1} \otimes x^k \\ &\quad + \frac{3\tilde{\lambda}k(k-1)}{4(2k+1)(2k-1)} \cdot f^{-1} \otimes x^{k-2}. \end{aligned}$$

From this formula it follows easily that $\mathfrak{L}^4$ maps $L(k)$ to $L(k+4) \oplus L(k) \oplus L(k-4)$ if $k \geq 2$, and to $L(k+2) \oplus L(k)$ for $k = 0, 1$, with non-zero projections to each summand. This proves the claim.

Remark 74. From the proof of Theorem 67, the combinatorics of tensoring with $L(1)$ can be recorded as follows:

$$\begin{aligned} L(1) \otimes V'(0, \tilde{\lambda}) &\cong V(1, \tilde{\lambda}); \\ L(1) \otimes V'(2, \tilde{\lambda}) &\cong V(1, \tilde{\lambda}); \\ L(1) \otimes V(1, \tilde{\lambda}) &\cong V'(0, \tilde{\lambda}) \oplus V'(2, \tilde{\lambda}) \oplus V(2, \tilde{\lambda}); \\ L(1) \otimes V(i, \tilde{\lambda}) &\cong V(i-1, \tilde{\lambda}) \oplus V(i+1, \tilde{\lambda}), \quad i > 1. \end{aligned}$$

In particular, the additive closure of $V'(0, \tilde{\lambda})$, $V'(2, \tilde{\lambda})$ and all $V(i, \tilde{\lambda})$ forms a simple $\mathcal{F}$ -module category, cf. Propositions 38 and 58. From the above formulae, we see that the combinatorics of this $\mathcal{F}$ -module category is different from the combinatorics of the $\mathcal{F}$ -module categories described in Propositions 38 and 58.

We note that there might exist, potentially, other simple $\mathfrak{g}$ -Harish-Chandra modules for $\mathfrak{L}^4$ which correspond to characters of $\text{Sym}(\tau)^{\mathfrak{g}}$ that do not occur in simple highest weight modules. The problem here seems to be the absence, in case of $\mathfrak{L}^4$, of an analogue of the classical theorem by Harish-Chandra that, in case of reductive Lie algebras, any central character is realizable on some highest weight module.

7.4. Some speculations in case of the Lie algebras $\mathfrak{L}^n = \mathfrak{sl}_2 \ltimes L(n)$ with n even. For an even non-negative integer n , consider the Lie algebra $\mathfrak{L}^n = \mathfrak{sl}_2 \ltimes L(n)$. The algebra $\mathfrak{L}^0$ is reductive and hence all its $\mathfrak{sl}_2$ -Harish-Chandra modules are finite-dimensional. Classification of all $\mathfrak{sl}_2$ -Harish-Chandra modules for the Takiff Lie algebra $\mathfrak{L}^2$ is given in Section 4. Note that all simple $\mathfrak{sl}_2$ -Harish-Chandra modules for $\mathfrak{L}^2$ are connected to highest weight $\mathfrak{L}^2$ -modules. For the Lie algebra $\mathfrak{L}^4$, all $\mathfrak{sl}_2$ -Harish-Chandra modules that are connected to highest weight $\mathfrak{L}^4$ -modules are classified in the previous subsection. Potentially, these are not all $\mathfrak{sl}_2$ -Harish-Chandra modules for $\mathfrak{L}^4$. As we see, the level of difficulty of the problem increases drastically with n .

The highest weight theory for $\mathfrak{L}^n$ is described in [LMZ14, Theorem 4]. As vector space, simple highest weight $\mathfrak{L}^n$ -modules look the same, independently of n . Therefore we expect that the problem of classification of simple $\mathfrak{sl}_2$ -Harish-Chandra modules for $\mathfrak{L}^n$ that are connected to highest weight $\mathfrak{L}^n$ -modules via Enright-Arkhipov functor should be solvable. One of the crucial missing ingredients, at the moment, seems to be an analogue of [HHLS18, Corollary 3.4] for general n (i.e., for general k in the notation of [HHLS18]).

Of special interest is the question of what kind of a monoidal representation of the monoidal category of finite-dimensional $\mathfrak{sl}_2$ -modules do the $\mathfrak{L}^n$ -modules from the previous paragraph form.

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