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Adedji, Kouessi Norbert; Filipin, Alan; Togbe, Alain

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3 authors:

Kouèssi Norbert Adédji
Institute of Mathematics and Physics

Alan Filipin
University of Zagreb

Alain Togbé
Purdue University Northwest

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Fibonacci and Lucas numbers as products of three repdgits in base *g*

Kouèssi Norbert Adédji[1] · Alan Filipin[2] · Alain Togbé[3]

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Abstract
Recall that a repdigit in base g is a positive integer that has only one digit in its base g expansion; i.e., a number of the form $a(g^m - 1)/(g - 1)$, for some positive integers $m \geq 1$, $g \geq 2$ and $1 \leq a \leq g-1$. In the present study, we investigate all Fibonacci or Lucas numbers which are expressed as products of three repdigits in base g. As illustration, we consider the case $g = 10$ where we show that the numbers 144 and 18 are the largest Fibonacci and Lucas numbers which can be expressible as products of three repdigits respectively. All this is done using linear forms in logarithms of algebraic numbers.

Keywords Fibonacci numbers · Lucas numbers · Mersenne numbers · Diophantine equations · g Repdigit · Linear forms in logarithms · Reduction method

Mathematics Subject Classification 11B39 · 11J86 · 11D61 · 11D72 · 11Y50

1 Introduction

Let $\{F_n\}_{n\geq 0}$ be the Fibonacci sequence given by $F_{n+2} = F_{n+1} + F_n$, with initial values $F_0 = 0$ and $F_1 = 1$ and let $\{L_n\}_{n\geq 0}$ be the Lucas sequence defined by $L_{n+2} = L_{n+1} + L_n$,

✉ Alain Togbé
atogbe@pnw.edu

Kouèssi Norbert Adédji
adedjnorb1988@gmail.com

Alan Filipin
filipin@grad.hr

1 Institut de Mathématiques et de Sciences Physiques, Université D'Abomey-Calavi, Abomey Calavi, Bénin

2 Faculty of Civil Engineering, University of Zagreb, Fra Andrije Kačića-Miošića 26, 10000 Zagreb, Croatia

3 Department of Mathematics and Statistics, Purdue University Northwest, 2200 169th Street, Hammond, IN 46323, USA

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where $L_0 = 2$ and $L_1 = 1$. If

$$(\alpha, \beta) = \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2} \right)$$

is the pair of roots of the characteristic equation $x^2 - x - 1 = 0$ of both the Fibonacci and Lucas numbers, then the Binet's formulas for their general terms are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n, \quad \text{for} \quad n \geq 0. \tag{1.1}$$

One can see that $1 < \alpha < 2$, $-1 < \beta < 0$ and $\alpha\beta = -1$. The following relations between the n-th Fibonacci number F_n, the n-th Lucas number L_n and α are well known

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1} \quad \text{and} \quad \alpha^{n-1} \leq L_n \leq 2\alpha^n, \quad \text{for} \quad n \geq 0. \tag{1.2}$$

Now, we are going to introduce the second concept of this study related to repdigits. Let $g \geq 2$ be an integer. A positive integer N is called a repdigit in base g or simply a g-repdigit if all of the digits in its base g expansion are equal. Indeed, N is of the form

$$a\left(\frac{g^m - 1}{g - 1}\right), \quad \text{for } m \geq 1, \ a \in \{1, 2, \ldots, g-1\}.$$

Taking $g = 10$, the positive integer N is simply called repdigit.

The study of Diophantine equations involving linear recurrent sequences and repdigits has been considered in recent years by many number theorists. First, Luca showed in [12] that the number 55 is the largest repdigit in the Fibonacci sequence. After this, many authors have worked on other similar problems (see [2, 5, 8, 9] and references therein). In [7], the authors solved the problem of finding the Fibonacci or Lucas numbers which are products of two repdigits. In [10], the authors tackled the problem of finding Fibonacci or Lucas numbers which are products of two repdigits in base b. In [1], we found all Padovan and Perrin numbers which are products of two repdigits in base b with $2 \leq b \leq 10$. This study deals with Fibonacci and Lucas numbers which are products of three repdigits in base g. In other words, we study the Diophantine equations

$$F_k = d_1\left(\frac{g^\ell - 1}{g - 1}\right) \cdot d_2\left(\frac{g^m - 1}{g - 1}\right) \cdot d_3\left(\frac{g^n - 1}{g - 1}\right), \tag{1.3}$$

and

$$L_k = d_1\left(\frac{g^\ell - 1}{g - 1}\right) \cdot d_2\left(\frac{g^m - 1}{g - 1}\right) \cdot d_3\left(\frac{g^n - 1}{g - 1}\right), \tag{1.4}$$

where $d_1, d_2, d_3, k, \ell, m$ and n are positive integers such that

$$1 \leq d_1, d_2, d_3 \leq g - 1 \text{ and } g \geq 2 \text{ with } n \geq 1, \ \ell \leq m \leq n. \tag{1.5}$$

The novelty of the present work lies in its effectiveness, in the sense that k and n can be effectively bounded in terms of g. This may be obtained using Baker's method. There are several different estimates of Baker-type lower bounds for linear forms in logarithms. In this study, we use the most common Baker type result due to Matveev [13] or [3, Theorem 9.4]. Thus, our main result is as follows.

Theorem 1.1 *Let $g \geq 2$ be an integer. Then, the Diophantine equations* (1.3) *and* (1.4) *have only finitely many solutions in integers $k, d_1, d_2, d_3, g, \ell, m, n$ such that* (1.5)*. Namely, we have*

$$\ell \leq m \leq n < 1.08 \times 10^{48} \log^9 g \quad \textit{and} \quad k < 1.08 \times 10^{49} \log^{10} g.$$

Remark 1.2 The inequalities from Theorem 1.1 allow one to compute upper bounds for all the solutions to (1.3) and (1.4), for every fixed g.

The organization of this paper is as follows. In Sect. 2, we will cite the results that we will use in Sects. 3 and 4, where our fundamental results of this paper will be fully proven. Also, we devote Sect. 5 to some concluding remarks.

2 Auxiliary results

We begin this section with a few reminders about the logarithmic height of an algebraic number. Let η be an algebraic number of degree d, let $a_0 > 0$ be the leading coefficient of its minimal polynomial over $\mathbb{Z}$ and let $\eta = \eta^{(1)}, \ldots, \eta^{(d)}$ denote its conjugates. The quantity defined by

$$h(\eta) = \frac{1}{d}\left(\log|a_0| + \sum_{j=1}^{d} \log\max\left(1, \left|\eta^{(j)}\right|\right)\right)$$

is called the logarithmic height of η. Some properties of height are as follows. For η_1, η_2 algebraic numbers and $m \in \mathbb{Z}$, we have

$$\begin{aligned} h(\eta_1 \pm \eta_2) &\leq h(\eta_1) + h(\eta_2) + \log 2, \\ h(\eta_1 \eta_2^{\pm}) &\leq h(\eta_1) + h(\eta_2), \\ h(\eta_1^m) &= |m| h(\eta_1). \end{aligned}$$

If $\eta = \dfrac{p}{q} \in \mathbb{Q}$ is a rational number in reduced form with $q \geq 1$, then the above definition becomes $h(\eta) = \log(\max\{|p|, q\})$. We can now present the famous Matveev's result used in this study. Thus, let $\mathbb{L}$ be a real number field of degree $d_{\mathbb{L}}$, $\eta_1, \ldots, \eta_s \in \mathbb{L}$ and $b_1, \ldots, b_s \in \mathbb{Z} \setminus \{0\}$. Let $B \geq \max\{|b_1|, \ldots, |b_s|\}$ and

$$\Lambda = \eta_1^{b_1} \cdots \eta_s^{b_s} - 1.$$

Let $A_1, \ldots, A_s$ be real numbers with

$$A_i \geq \max\{d_{\mathbb{L}} h(\eta_i), |\log \eta_i|, 0.16\}, \quad i = 1, 2, \ldots, s.$$

With the above notations, Matveev proved the following result (the version that we will use is due to Bugeaud, Mignotte and Siksek. See Theorem 9.4 in [3]).

Theorem 2.1 *Assume that $\Lambda \neq 0$. Then*

$$\log|\Lambda| > -1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot d_{\mathbb{L}}^2 \cdot (1 + \log d_{\mathbb{L}}) \cdot (1 + \log B) \cdot A_1 \cdots A_s.$$

The following lemma will also be used in order to prove our subsequent results.

Lemma 2.2 ([11, Lemma 7]) *If* $l \geq 1$, $H > \left(4l^2\right)^l$ *and* $H > L/(\log L)^l$, *then*

$$L < 2^l H(\log H)^l.$$

The upper bounds of the variables of Eqs. (1.3) and (1.4) obtained after the application of Theorem 2.1 are too large for a very fast search for solutions by a computer program. To overcome this situation, a reduction of the upper bounds is necessary. For this reduction's purpose, we present a variant of the reduction method of Baker and Davenport due to Dujella and Pethő [6]. For a real number x, we write $||x|| := \min\{|x-n| : n \in \mathbb{Z}\}$ for the distance from x to the nearest integer.

Lemma 2.3 ([6, Lemma 5a]) *Let M be a positive integer, let p/q be a convergent of the continued fraction of the irrational τ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let*

$$\varepsilon = ||\mu q|| - M \cdot ||\tau q||,$$

where $||\cdot||$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there is no solution of the inequality

$$0 < |m\tau - n + \mu| < AB^{-w}$$

in positive integers m, n and w with

$$m \leq M \quad and \quad w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

Note that Lemma 2.3 cannot be applied when $\mu = 0$ (since then $\varepsilon < 0$) or when μ is a multiple of τ. For this case, we use the following well known technical result from Diophantine approximation, known as Legendre's criterion.

Lemma 2.4 (see [4]) *Let κ be a real number and x, y integers such that*

$$\left|\kappa - \frac{x}{y}\right| < \frac{1}{2y^2}.$$

Then $x/y = p_k/q_k$ is a convergent of the continued fraction expansion $[a_0, a_1, \ldots]$ of κ (with some $k = 0, 1, \ldots$). Further, let M and N be nonnegative integers such that $q_N > M$. Then putting $a(M) := \max\{a_i : i = 0, 1, \ldots, N\}$, the inequality

$$\left|\kappa - \frac{x}{y}\right| \geq \frac{1}{(a(M)+2)y^2},$$

holds for all pairs (x, y) of positive integers with $0 < y < M$.

3 Fibonacci numbers as products of three repdgits in base *g*

Our first aim is to prove Theorem 1.1 using the Diophantine equation (1.3).

3.1 Proof of Theorem 1.1 for Eq. (1.3)

Note that if $n = 1$, then $\ell = m = 1$ and therefore the Diophantine equation (1.3) becomes $F_k = d_1 d_2 d_3$. Combining this with (1.2), we have $k \leq 3\log(g-1)/\log\alpha + 2$. So, in this case the bound of k from Theorem 1.1 easily holds. For the rest of the proof we consider $n \geq 2$. The following result will be useful in proving our main result which gives a relation between n and k in Eq. (1.3).

Lemma 3.1 *All solutions of the Diophantine equation (1.3) satisfy*

$$k < 3n\frac{\log g}{\log \alpha} + 2 < 10n \log g.$$

Proof From (1.2), we have

$$\alpha^{k-2} \leq F_k = d_1\left(\frac{g^\ell - 1}{g-1}\right) \cdot d_2\left(\frac{g^m - 1}{g-1}\right) \cdot d_3\left(\frac{g^n - 1}{g-1}\right) \leq (g^n - 1)^3 < g^{3n}.$$

Taking logarithm on both sides, we get $(k-2)\log \alpha < 3n \log g$. Since, $n \geq 2$ and $g \geq 2$, we obtain the desired inequalities. This ends the proof. □

Next, we find upper bounds for the variables n, ℓ, m of Eq. (1.3). Using (1.1) and (1.3), we get

$$F_k = \frac{\alpha^k}{\sqrt{5}} - \frac{\beta^k}{\sqrt{5}} = d_1\left(\frac{g^\ell - 1}{g-1}\right) \cdot d_2\left(\frac{g^m - 1}{g-1}\right) \cdot d_3\left(\frac{g^n - 1}{g-1}\right)$$

and thus obtain

$$\begin{aligned}\frac{\alpha^k}{\sqrt{5}} - \frac{d_1 d_2 d_3 g^{\ell+m+n}}{(g-1)^3} &= \frac{\beta^k}{\sqrt{5}} - \frac{d_1 d_2 d_3 g^{\ell+m}}{(g-1)^3} - \frac{d_1 d_2 d_3 g^{n+\ell}}{(g-1)^3} + \frac{d_1 d_2 d_3 g^{\ell}}{(g-1)^3} \\ &\quad - \frac{d_1 d_2 d_3 g^{n+m}}{(g-1)^3} + \frac{d_1 d_2 d_3 g^{m}}{(g-1)^3} + \frac{d_1 d_2 d_3 g^{n}}{(g-1)^3} - \frac{d_1 d_2 d_3}{(g-1)^3}. \qquad (3.1)\end{aligned}$$

Taking the absolute values of both sides of (3.1), we get

$$\begin{aligned}\left|\frac{\alpha^k}{\sqrt{5}} - \frac{d_1 d_2 d_3 g^{\ell+m+n}}{(g-1)^3}\right| &< \frac{1}{\alpha^k\sqrt{5}} + \frac{d_1 d_2 d_3 g^{\ell+m}}{(g-1)^3} + \frac{d_1 d_2 d_3 g^{n+\ell}}{(g-1)^3} + \frac{d_1 d_2 d_3 g^{\ell}}{(g-1)^3} \\ &\quad + \frac{d_1 d_2 d_3 g^{n+m}}{(g-1)^3} + \frac{d_1 d_2 d_3 g^{m}}{(g-1)^3} + \frac{d_1 d_2 d_3 g^{n}}{(g-1)^3} + \frac{d_1 d_2 d_3}{(g-1)^3}. \qquad (3.2)\end{aligned}$$

Dividing both sides of (3.2) by $\frac{d_1 d_2 d_3 g^{\ell+n+m}}{(g-1)^3}$ and using the fact that $\ell \leq m \leq n$ gives us the following inequalities

$$\begin{aligned}\left|\frac{(g-1)^3 \cdot \alpha^k \cdot g^{-(\ell+n+m)}}{d_1 d_2 d_3\sqrt{5}} - 1\right| &< \frac{(g-1)^3}{\alpha^k d_1 d_2 d_3 g^{\ell+n+m}\sqrt{5}} + \frac{1}{g^n} + \frac{1}{g^m} + \frac{1}{g^{n+m}} \\ &\quad + \frac{1}{g^\ell} + \frac{1}{g^{n+\ell}} + \frac{1}{g^{\ell+m}} + \frac{1}{g^{\ell+m+n}} \\ &< 8 \cdot g^{-\ell}.\end{aligned}$$

From this, it follows that

$$\left|\frac{(g-1)^3}{d_1 d_2 d_3\sqrt{5}} \cdot \alpha^k \cdot g^{-(\ell+n+m)} - 1\right| < \frac{8}{g^\ell}. \qquad (3.3)$$

Put

$$\Lambda_1 := \frac{(g-1)^3}{d_1 d_2 d_3\sqrt{5}} \cdot \alpha^k \cdot g^{-(\ell+n+m)} - 1.$$

Let us show that $\Lambda_1 \neq 0$. Suppose $\Lambda_1 = 0$, then

$$\alpha^{2k} = \frac{5(d_1d_2d_3)^2}{(g-1)^6} \cdot g^{2(\ell+m+n)},$$

which is false as α^{2k} cannot be rational except if $k = 0$. Thus, $\Lambda_1 \neq 0$. To apply Theorem 2.1 to Λ_1, we choose the following data

$$(\eta_1, b_1) := \left(\frac{(g-1)^3}{d_1d_2d_3\sqrt{5}}, 1\right), \ (\eta_2, b_2) := (\alpha, k), \ (\eta_3, b_3) := (g, -(\ell+m+n))$$

and $s := 3$. Note that $\eta_1, \eta_2, \eta_3 \in \mathbb{Q}(\alpha)$ and $b_1, b_2, b_3 \in \mathbb{Z}$. The degree $d_{\mathbb{L}} = [\mathbb{Q}(\alpha) : \mathbb{Q}] = 2$, where $\mathbb{L}$ is $\mathbb{Q}(\alpha)$. According to the inequalities from Lemma 3.1, we can take $B := 10n \log g$ because $B \geq \max\{|b_1|, |b_2|, |b_3|\}$. To estimate the parameters A_1, A_2, A_3, we calculate the logarithmic heights of η_1, η_2, η_3 as follows:

$$\begin{aligned} h(\eta_1) = h\left(\frac{(g-1)^3}{d_1d_2d_3\sqrt{5}}\right) &\leq h\left(\frac{(g-1)^3}{d_1d_2d_3}\right) + h\left(\frac{1}{\sqrt{5}}\right) \\ &= \log\left(\max\{(g-1)^3, d_1d_2d_3\}\right) + \frac{1}{2}\log 5 \\ &= 3\log(g-1) + \frac{1}{2}\log 5 < 4\log g \end{aligned}$$

and

$$h(\alpha) = \frac{1}{2}\log\alpha \quad \text{and} \quad h(\eta_3) = \log g.$$

Thus, one can take

$$A_1 = 8\log g, \quad A_2 = \log\alpha \quad \text{and} \quad A_3 = 2\log g.$$

Then, we apply Theorem 2.1 and find

$$\log|\Lambda_1| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 4 \cdot (1+\log 2) \cdot (1+\log(10n\log g)) \cdot A, \tag{3.4}$$

where $A := A_1A_2A_3 = 16\log\alpha \cdot \log^2 g$. Comparing inequality (3.4) with (3.3) gives

$$\ell\log g - \log 8 < 7.47 \times 10^{12} \cdot (1+\log(10n\log g)) \cdot \log^2 g. \tag{3.5}$$

Because for $g \geq 2$ and $n \geq 2$,

$$1 + \log(10n\log g) < 10\log n \cdot \log g,$$

we get

$$\ell < 7.5 \times 10^{13} \cdot \log n \cdot \log^2 g. \tag{3.6}$$

Secondly, we rewrite (1.3) as

$$\frac{\alpha^k(g-1)}{d_1(g^\ell-1)\sqrt{5}} - \frac{\beta^k(g-1)}{d_1(g^\ell-1)\sqrt{5}} = \frac{d_2d_3}{(g-1)^2}\left(g^{n+m} - g^m - g^n + 1\right),$$

which implies

$$\begin{aligned} \frac{\alpha^k(g-1)}{d_1(g^\ell-1)\sqrt{5}} - \frac{d_2d_3g^{n+m}}{(g-1)^2} &= \frac{\beta^k(g-1)}{d_1(g^\ell-1)\sqrt{5}} - \frac{d_2d_3g^m}{(g-1)^2} - \frac{d_2d_3g^n}{(g-1)^2} \\ &\quad + \frac{d_2d_3}{(g-1)^2}. \end{aligned} \tag{3.7}$$

Taking the absolute values of both sides of (3.7), we have

$$\left|\frac{\alpha^k(g-1)}{d_1(g^\ell-1)\sqrt{5}} - \frac{d_2d_3g^{n+m}}{(g-1)^2}\right| < \frac{(g-1)}{d_1(g^\ell-1)\alpha^k\sqrt{5}} + \frac{d_2d_3g^m}{(g-1)^2} + \frac{d_2d_3g^n}{(g-1)^2} + \frac{d_2d_3}{(g-1)^2}.$$

Dividing both sides of the inequality above by $\frac{d_2d_3g^{n+m}}{(g-1)^2}$ and using the fact that $n \geq 2$, leads to

$$\left|\frac{(g-1)^3}{d_1d_2d_3(g^\ell-1)\sqrt{5}} \cdot \alpha^k \cdot g^{-(n+m)} - 1\right| < \frac{(g-1)^3}{d_1d_2d_3(g^\ell-1)\alpha^k g^{n+m}\sqrt{5}} + \frac{1}{g^n} + \frac{1}{g^m} + \frac{1}{g^{n+m}} < 4 \cdot g^{-m}.$$

Therefore, we obtain

$$\left|\frac{(g-1)^3}{d_1d_2d_3(g^\ell-1)\sqrt{5}} \cdot \alpha^k \cdot g^{-(n+m)} - 1\right| < \frac{4}{g^m}. \tag{3.8}$$

Now, let us apply Theorem 2.1 with

$$(\eta_1, b_1) := \left(\frac{(g-1)^3}{d_1d_2d_3(g^\ell-1)\sqrt{5}}, 1\right), (\eta_3, b_3) := (g, -n-m),$$

$(\eta_2, b_2) := (\alpha, k)$, $s := 3$ and $B := 10n \log g$. Note that the numbers η_1, η_2, and η_3 are positive real numbers and elements of the field $\mathbb{L} = \mathbb{Q}(\sqrt{5})$. It is obvious that the degree of the field $\mathbb{L}$ is 2. So $d_{\mathbb{L}} = 2$. Let

$$\Lambda_2 := \frac{(g-1)^3}{d_1d_2d_3(g^\ell-1)\sqrt{5}} \cdot \alpha^k \cdot g^{-(n+m)} - 1.$$

If $\Lambda_2 = 0$, then we get

$$\alpha^{2k} = \frac{5(d_1d_2d_3)^2(g^\ell-1)^2g^{2(n+m)}}{(g-1)^6} \in \mathbb{Q}.$$

This is impossible as α^{2k} is irrational for $k \geq 1$. Therefore, Λ_2 is nonzero. Moreover, since

$$\begin{aligned} h(\eta_1) = h\left(\frac{(g-1)^3}{d_1d_2d_3(g^\ell-1)\sqrt{5}}\right) &\leq h\left(\frac{(g-1)^3}{d_1d_2d_3}\right) + h\left(\frac{1}{g^\ell-1}\right) + h\left(\frac{1}{\sqrt{5}}\right) \\ &= \log\max\{(g-1)^3, d_1d_2d_3\} + \log(g^\ell-1) + \frac{1}{2}\log 5 \\ &= 3\log(g-1) + \log(g^\ell-1) + \frac{1}{2}\log 5 < (5+\ell)\log g, \end{aligned}$$

and

$$h(\eta_2) = \frac{1}{2}\log\alpha, \quad h(\eta_3) = \log g,$$

we can take $A_1 := 2(\ell+5)\log g$, $A_2 := \log\alpha$, and $A_3 := 2\log g$. Thus, taking into account the inequality (3.8) and using Theorem 2.1, we obtain

$$m\log g - \log 4 < 1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 4 \cdot (1+\log 2) \cdot (1+\log B) \cdot A,$$

where $A = A_1A_2A_3 = 4(5+\ell)\log\alpha\cdot\log^2 g$. Since $g \geq 2$ and $1+\log B < 10\log n\log g$, it follows that

$$m < 1.87\times 10^{13}\cdot(5+\ell)\cdot\log n\cdot\log^2 g + 2. \tag{3.9}$$

By (3.6), we have

$$5+\ell < 7.6\times 10^{13}\cdot\log n\cdot\log^2 g. \tag{3.10}$$

Therefore, from (3.9) and (3.10), we easily get

$$m < 1.5\times 10^{27}\log^2 n\log^4 g. \tag{3.11}$$

Rearranging now Eq. (1.3) as

$$\frac{d_3g^n}{g-1} - \frac{(g-1)^2\alpha^k}{d_1d_2(g^\ell-1)(g^m-1)\sqrt{5}} = \frac{d_3}{g-1} - \frac{\beta^k(g-1)^2}{d_1d_2(g^\ell-1)(g^m-1)\sqrt{5}}$$

and taking absolute values of both sides of the equality above, we get

$$\left|\frac{d_3g^n}{g-1} - \frac{(g-1)^2\alpha^k}{d_1d_2(g^\ell-1)(g^m-1)\sqrt{5}}\right| < \frac{d_3}{g-1} + \frac{(g-1)^2}{\alpha^k d_1d_2(g^\ell-1)(g^m-1)\sqrt{5}}. \tag{3.12}$$

Dividing both sides of (3.12) by $\dfrac{d_3g^n}{g-1}$ and using the fact that $n \geq 2$, we obtain

$$\left|1 - \frac{(g-1)^3}{d_1d_2d_3(g^\ell-1)(g^m-1)\sqrt{5}}\cdot g^{-n}\cdot\alpha^k\right| < \frac{1}{g^n} + \frac{1}{g^{n-1}} < \frac{2}{g^{n-1}}. \tag{3.13}$$

Put

$$\Lambda_3 := \frac{(g-1)^3}{d_1d_2d_3(g^\ell-1)(g^m-1)\sqrt{5}}\cdot g^{-n}\cdot\alpha^k - 1. \tag{3.14}$$

Next, we apply Theorem 2.1 to (3.14). First, we need to check that $\Lambda_3 \neq 0$. If it is, then we would get

$$\alpha^{2k} = \frac{5(d_1d_2d_3)^2(g^\ell-1)^2(g^m-1)^2g^{2n}}{(g-1)^6} \in \mathbb{Q},$$

which is impossible. Hence, $\Lambda_3 \neq 0$. So, we apply Theorem 2.1 to (3.14) with the data:

$$s := 3,\ (\eta_1, b_1) := \left(\frac{(g-1)^3}{d_1d_2d_3(g^\ell-1)(g^m-1)\sqrt{5}}, 1\right), (\eta_2, b_2) := (g, -n),$$

and $(\eta_3, b_3) := (\alpha, k)$. Because $B \geq \max\{|b_1|, |b_2|, |b_3|\} = \max\{1, n, k\}$ and by the inequality from Lemma 3.1, we see that we can take $B := 10n\log g$. Note that $\eta_1, \eta_2, \eta_3 \in \mathbb{Q}(\alpha)$. Observe that $\mathbb{L} := \mathbb{Q}(\eta_1, \eta_2, \eta_3) = \mathbb{Q}(\alpha)$, so $d_{\mathbb{L}} = 2$. Next,

$$\begin{aligned} h(\eta_1) &= h\left(\frac{(g-1)^3}{d_1d_2d_3(g^\ell-1)(g^m-1)\sqrt{5}}\right) \\ &\leq h\left(\frac{(g-1)^3}{d_1d_2d_3}\right) + h\left((g^\ell-1)(g^m-1)\right) + h(\sqrt{5}) \\ &< (3+\ell+m)\log g + \frac{1}{2}\log 5 < (5+\ell+m)\log g. \end{aligned}$$

Thus, we take

$$A_1 := 2(5+\ell+m)\log g, \quad A_2 := 2\log g \quad \text{and} \quad A_3 := \log\alpha.$$

Theorem 2.1 gives

$$\log|\Lambda_3| > -1.4\cdot 30^6\cdot 3^{4.5}\cdot 4\cdot(1+\log 2)\cdot(1+\log B)\cdot A \tag{3.15}$$

with

$$A = A_1A_2A_3 = 2\log g\cdot\log\alpha\cdot(10+2\ell+2m)\log g \tag{3.16}$$

and

$$1+\log B < 10\log n\log g. \tag{3.17}$$

Combining the above three relations with (3.13) implies

$$(n-1)\log g - \log 2 < 1.87\times 10^{13}(5+\ell+m)\log n\log^3 g,$$

which leads to

$$n < 1.87\times 10^{13}(5+\ell+m)\log n\log^2 g + 2. \tag{3.18}$$

Referring to the relations (3.6) and (3.11), we have

$$\begin{aligned}5+\ell+m &< 5+7.5\times 10^{13}\cdot\log n\cdot\log^2 g + 1.5\times 10^{27}\log^2 n\log^4 g\\ &< 3\times 10^{27}\log^2 n\log^4 g.\end{aligned} \tag{3.19}$$

Inserting this in (3.18) leads to

$$n < 5.7\times 10^{40}\cdot\log^3 n\cdot\log^6 g. \tag{3.20}$$

We are now in position to apply Lemma 2.2 with the data

$$l=3, \quad L=n \quad \text{and} \quad H := 5.7\times 10^{40}\cdot\log^6 g.$$

Therefore, we get

$$\begin{aligned}n &< 2^3\cdot 5.7\times 10^{40}\cdot\log^6 g\times\log^3\left(5.7\times 10^{40}\cdot\log^6 g\right)\\ &< 2^3\cdot 5.7\times 10^{40}\cdot\log^6 g\cdot(93.84+6\log\log g)^3\\ &< 1.08\times 10^{48}\log^9 g.\end{aligned}$$

In the above inequality, we have used the fact that $93.84+6\log\log g < 133\log g$, which holds for all $g\geq 2$. Hence, we summarize that all the solutions of (1.3) satisfy

$$n < 1.08\times 10^{48}\log^9 g \quad \text{and} \quad k < 10n\log g < 1.08\times 10^{49}\log^{10} g.$$

The proof of Theorem 1.1 is complete in this case. □

3.2 Application for the decimal base

Now, as an illustration, we solve the Diophantine equation (1.3) for $g=10$. When $g=10$, the bound on k becomes

$$k < 4.6\times 10^{52}.$$

Thus, our main result in this case is the following.

Theorem 3.2 *The only Fibonacci numbers which are products of three repdigits are* $1, 2, 3, 5, 8, 21, 55,$ *and* 144.

Proof First, we must reduce the bounds on ℓ, m, n and k. Put

$$\begin{aligned} z_1 &:= \log(\Lambda_1 + 1) \\ &= k \log \alpha - (\ell + m + n) \log 10 + \log \left(\frac{729}{d_1 d_2 d_3 \sqrt{5}} \right). \end{aligned}$$

Inequality (3.3) can be written as

$$\left| e^{z_1} - 1 \right| < \frac{8}{10^\ell}.$$

Since $\ell \geq 1$, we have $|e^{z_1} - 1| < \frac{8}{10^\ell} \leq \frac{4}{5}$, which implies that $\frac{5}{9} < e^{-z_1} < 5$. If $z_1 > 0$, then

$$0 < z_1 < e^{z_1} - 1 = \left| e^{z_1} - 1 \right| < \frac{8}{10^\ell}.$$

Furthermore, if $z_1 < 0$, then

$$0 < |z_1| < e^{|z_1|} - 1 = e^{-z_1}(1 - e^{z_1}) < \frac{40}{10^\ell}.$$

In any case, it always holds $0 < |z_1| < \frac{40}{10^\ell}$, which implies

$$0 < \left| k \frac{\log \alpha}{\log 10} - (\ell + m + n) + \frac{\log \left(729/(d_1 d_2 d_3 \sqrt{5}) \right)}{\log 10} \right| < 17.4 \cdot 10^{-\ell}. \tag{3.21}$$

It is easy to see that $\frac{\log \alpha}{\log 10}$ is irrational. In fact, if $\frac{\log \alpha}{\log 10} = \frac{p}{q}$ ($p, q \in \mathbb{Z}$ and $p > 0, q > 0$, $\gcd(p, q) = 1$), then $10^p = \alpha^q \in \mathbb{Z}$ which is an absurdity. Now, we will apply Lemma 2.3 with $w := \ell$,

$$\tau := \frac{\log \alpha}{\log 10}, \quad \mu := \frac{\log \left(729/(d_1 d_2 d_3 \sqrt{5}) \right)}{\log 10}, \quad A := 17.4, \quad B := 10.$$

Because $k < 4.6 \times 10^{52}$, we can take $M := 4.6 \times 10^{52}$. Therefore, for the remaining proof, we use Mathematica to apply Lemma 2.3. For the computations, if the first convergent such that $q > 6M$ does not satisfy the condition $\varepsilon > 0$, then we use the next convergent until we find the one that satisfies the conditions. We have found that the denominator of the 115th convergent

$$\frac{p_{115}}{q_{115}} = \frac{153228251473297124869936026285513734768562420308679261 4}{733192887818698250118437049124929795282465913106280609 9}$$

of τ exceeds $6M$. Thus, we can say that the inequality (3.21) has no solution for

$$\ell = w \geq \frac{\log(A q_{115}/\varepsilon)}{\log 10} \geq \frac{\log(A q_{115}/0.00809526)}{\log 10} \geq 58.1975.$$

So, we obtain

$$\ell \leq 58. \tag{3.22}$$

Substituting this upper bound for ℓ into (3.9) and combining the new bound obtained with (3.18), we get

$$n < 6.4 \times 10^{29} \cdot \log^2 n,$$

which implies $n < 1.3 \times 10^{34}$ using Lemma 2.2. Thus, by Lemma 3.1 we have $k < 3 \times 10^{35}$. Next, we need to reduce the bound on m. We return to (3.8) and put

$$\begin{aligned} z_2 &:= \log(\Lambda_2 + 1) \\ &= k \log \alpha - (n+m)\log 10 + \log\left(\frac{729}{d_1 d_2 d_3 (10^\ell - 1)\sqrt{5}}\right). \end{aligned}$$

From the inequality (3.8) and $m \geq 1$, we conclude that

$$\left|e^{z_2} - 1\right| < \frac{4}{10^m} < \frac{1}{2},$$

which implies that $\frac{1}{2} < e^{z_2} < \frac{3}{2}$. If $z_2 > 0$, then $0 < z_2 < e^{z_2} - 1 < \frac{4}{10^m}$. If $z_2 < 0$, then

$$0 < |z_2| < e^{|z_2|} - 1 = e^{-z_2} - 1 = e^{-z_2}(1 - e^{z_2}) < \frac{8}{10^m}.$$

In any case, we have $0 < |z_2| < \frac{8}{10^m}$, which implies

$$0 < \left| k \frac{\log \alpha}{\log 10} - (n+m) + \frac{\log\left(729/(d_1 d_2 d_3 (10^\ell - 1)\sqrt{5})\right)}{\log 10} \right| < \frac{3.5}{10^m}. \tag{3.23}$$

Again, we apply Lemma 2.3 with

$$\tau := \frac{\log \alpha}{\log 10}, \;\; \mu := \frac{\log\left(729/(d_1 d_2 d_3 (10^\ell - 1)\sqrt{5})\right)}{\log 10}, \;\; A := 3.5, \;\; B := 10$$

and $M := 3 \times 10^{35}$. With the help of Mathematica, we found that the denominator of the 77th convergent

$$\frac{p_{77}}{q_{77}} = \frac{1097876139463713781430275039172749779}{5253306550332349137376600680873772748}$$

of τ exceeds $6M$. It follows that the inequality (3.23) has no solution for

$$m = w \geq \frac{\log(Aq_{77}/\varepsilon)}{\log 10} \geq \frac{\log(Aq_{77}/0.000111931)}{\log 10} \geq 41.2155.$$

Hence, we obtain

$$m \leq 41. \tag{3.24}$$

Inserting the bounds from (3.22) and (3.24) in (3.18), we get

$$n < 1.1 \times 10^{16} \log n + 2,$$

which leads to $n < 8.2 \times 10^{17}$ and then $k < 2 \times 10^{19}$. Finally, we have to reduce the bound on n. From (3.13), we can put

$$\begin{aligned} z_3 &:= \log(\Lambda_3 + 1) \\ &= k\log\alpha - n\log 10 + \log\left(\frac{729}{d_1 d_2 d_3 (10^\ell - 1)(10^m - 1)\sqrt{5}}\right). \end{aligned}$$

By following what is done in previous cases, it is easy to see that for $n \geq 2$, we have

$$0 < \left| k\frac{\log\alpha}{\log 10} - n + \frac{\log\left(729/\left(d_1 d_2 d_3 (10^\ell - 1)(10^m - 1)\sqrt{5}\right)\right)}{\log 10} \right| < \frac{1.8}{10^{n-1}}. \tag{3.25}$$

Now, we apply Lemma 2.3 to (3.25) with $B := 10$,

$$\tau := \frac{\log\alpha}{\log 10}, \quad \mu := \frac{\log\left(729/\left(d_1 d_2 d_3 (10^\ell - 1)(10^m - 1)\sqrt{5}\right)\right)}{\log 10}, \quad A := 1.8,$$

and $M := 2 \times 10^{19}$. Find upper bounds that the denominator of the 44th convergent

$$\frac{p_{44}}{q_{44}} = \frac{259791952914951895804}{1243097211893507332887}$$

of τ exceeds $6M$. Therefore the inequality (3.25) has no solution for

$$n - 1 = w \geq \frac{\log(Aq_{44}/\varepsilon)}{\log 10} \geq \frac{\log(Aq_{44}/0.0000637147)}{\log 10} \geq 25.5455.$$

Hence, we obtain

$$n \leq 26. \tag{3.26}$$

So, it remains to check Eq. (1.3) in the case $g = 10$ for $1 \leq d_1, d_2, d_3 \leq 9$, $1 \leq n \leq 26$, $1 \leq k \leq 598$, $1 \leq \ell \leq 58$ and $1 \leq m \leq 41$. A quick inspection using Maple reveals that the Diophantine equation (1.3) has only the solutions listed in the statement of Theorem 3.2, which ends the proof of Theorem 3.2. □

4 Lucas numbers as products of three repdigits in base *g*

In this section, we will follow the method from Sect. 3. To avoid repetition in this section, some details will be left out and Sect. 3 may be referred for any required clarifications. Our first aim is to prove Theorem 1.1 considering Eq. (1.4). The computations of the heights and the proof of the non-nullity of the linear forms on logarithms are similar. So, we leave them to the readers.

4.1 Proof of Theorem 1.1 for Eq. (1.4)

Again, by taking $n = 1$, we easily verify that the bound of k from Theorem 1.1 is valid. Now let us see what happens for $n \geq 2$. The following result will be useful in proving our main result which gives a relation between n and k of Eq. (1.4).

Lemma 4.1 *All solutions of the Diophantine equation (1.4) satisfy*

$$k < 3n\frac{\log g}{\log \alpha} + 1 < 10n\log g.$$

Proof The proof of this lemma is similar to that of Lemma 3.1. In this case, we have to combine (1.2) and (1.4). □

First, we find upper bounds for the variables n, ℓ, m of the Diophantine equation (1.4). Combining (1.1) and (1.4), we get

$$\alpha^k - \frac{d_1d_2d_3g^{\ell+m+n}}{(g-1)^3} = -\beta^k - \frac{d_1d_2d_3g^{\ell+m}}{(g-1)^3} - \frac{d_1d_2d_3g^{n+\ell}}{(g-1)^3} + \frac{d_1d_2d_3g^{\ell}}{(g-1)^3} - \frac{d_1d_2d_3g^{n+m}}{(g-1)^3} + \frac{d_1d_2d_3g^{m}}{(g-1)^3} + \frac{d_1d_2d_3g^{n}}{(g-1)^3} - \frac{d_1d_2d_3}{(g-1)^3} \tag{4.1}$$

and then

$$|\Lambda_4| := \left|\frac{(g-1)^3}{d_1d_2d_3}\cdot\alpha^k\cdot g^{-(\ell+n+m)} - 1\right| < \frac{8}{g^{\ell}}. \tag{4.2}$$

Next, we apply Theorem 2.1 to (4.2) and we get that

$$\ell < 5.7\times 10^{13}\cdot\log n\cdot\log^2 g. \tag{4.3}$$

Now, we rewrite (1.4) into the form

$$\frac{\alpha^k(g-1)}{d_1(g^{\ell}-1)} - \frac{d_2d_3g^{n+m}}{(g-1)^2} = -\frac{\beta^k(g-1)}{d_1(g^{\ell}-1)} - \frac{d_2d_3g^{m}}{(g-1)^2} - \frac{d_2d_3g^{n}}{(g-1)^2} + \frac{d_2d_3}{(g-1)^2}, \tag{4.4}$$

which leads us to

$$|\Lambda_5| := \left|\frac{(g-1)^3}{d_1d_2d_3(g^{\ell}-1)}\cdot\alpha^k\cdot g^{-(n+m)} - 1\right| < \frac{4}{g^{m}}. \tag{4.5}$$

We apply Theorem 2.1 to (4.5) and we see that

$$m < 1.87\times 10^{13}\cdot(3+\ell)\cdot\log n\cdot\log^2 g + 2. \tag{4.6}$$

By (4.3), we have

$$3+\ell < 5.8\times 10^{13}\cdot\log n\cdot\log^2 g. \tag{4.7}$$

Therefore, from (4.6) and (4.7), we easily get

$$m < 1.1\times 10^{27}\log^2 n\log^4 g. \tag{4.8}$$

Finally, rearranging now Eq. (1.4) as

$$\frac{d_3g^n}{g-1} - \frac{(g-1)^2\alpha^k}{d_1d_2(g^{\ell}-1)(g^m-1)} = \frac{d_3}{g-1} + \frac{\beta^k(g-1)^2}{d_1d_2(g^{\ell}-1)(g^m-1)}$$

and using the fact that $n \geq 2$, we obtain

$$|\Lambda_6| := \left|1 - \frac{(g-1)^3}{d_1d_2d_3(g^{\ell}-1)(g^m-1)}\cdot g^{-n}\cdot\alpha^k\right| < \frac{2}{g^{n-1}}. \tag{4.9}$$

Thus, with the help of Theorem 2.1 we have

$$n < 1.87 \times 10^{13}(3+\ell+m)\log n \log^2 g + 2. \tag{4.10}$$

Referring to the inequalities (4.3) and (4.8), we have

$$\begin{aligned} 3+\ell+m &< 5.8 \times 10^{13} \cdot \log n \cdot \log^2 g + 1.1 \times 10^{27} \log^2 n \log^4 g \\ &< 2.2 \times 10^{27} \log^2 n \log^4 g. \end{aligned} \tag{4.11}$$

Inserting this in (4.10) leads to

$$n < 4.2 \times 10^{40} \cdot \log^3 n \cdot \log^6 g. \tag{4.12}$$

Now, Lemma 2.2 tells us that

$$\begin{aligned} n &< 2^3 \cdot 4.2 \times 10^{40} \cdot \log^6 g \times \log^3 \left(4.2 \times 10^{40} \cdot \log^6 g\right) \\ &< 2^3 \cdot 4.2 \times 10^{40} \cdot \log^6 g \cdot (93.6 + 6 \log\log g)^3 \\ &< 7.73 \times 10^{47} \log^9 g. \end{aligned}$$

In the above inequality, we have used the fact that $93.6 + 6\log\log g < 132 \log g$, which holds for all $g \geq 2$. Hence, we summarize that all the solutions of (1.4) satisfy

$$n < 7.73 \times 10^{47} \log^9 g \quad \text{and} \quad k < 10n \log g < 7.73 \times 10^{48} \log^{10} g.$$

This finishes the proof of Theorem 1.1 in this case. □

4.2 Application for the decimal base

Again, as an illustration, we solve Eq. (1.4) for $g = 10$. Note that $k < 4.6 \times 10^{52}$. Here is our main result in this case.

Theorem 4.2 *The only Lucas numbers which are products of three repdigits are* 1, 3, 4, 7, 11 *and* 18.

Proof We must first reduce the bounds on ℓ, m, n and k. Put

$$z_4 := \log(\Lambda_4 + 1) = k \log\alpha - (\ell+m+n)\log 10 + \log\left(\frac{729}{d_1 d_2 d_3}\right).$$

From (4.2), we deduce that

$$0 < \left| k\frac{\log\alpha}{\log 10} - (\ell+m+n) + \frac{\log(729/(d_1 d_2 d_3))}{\log 10} \right| < 17.4 \cdot 10^{-\ell}. \tag{4.13}$$

Now, we have to study the following two cases.

Case 1: $(d_1, d_2, d_3) \neq (9,9,9)$. We apply Lemma 2.3 with $w := \ell$,

$$\tau := \frac{\log\alpha}{\log 10}, \quad \mu := \frac{\log(729/(d_1 d_2 d_3))}{\log 10}, \quad A := 17.4, \quad B := 10.$$

Because $k < 4.6 \times 10^{52}$, we can take $M := 4.6 \times 10^{52}$. We use Mathematica to apply Lemma 2.3 and found that the denominator of the 114th convergent

$$\frac{p_{114}}{q_{114}} = \frac{75199708224715672236920162770429633212096962359234385}{359828495765425172949832316042466402419242364862312251}$$

of τ exceeds $6M$. Thus, we can conclude that the inequality (4.13) has no solution for

$$\ell = w \geq \frac{\log(Aq_{114}/\varepsilon)}{\log 10} \geq \frac{\log(Aq_{114}/0.00114865)}{\log 10} \geq 57.7365.$$

Hence, we obtain

$$\ell \leq 57. \tag{4.14}$$

Case 2: $(d_1, d_2, d_3) = (9, 9, 9)$. In this case from (4.13), we have

$$0 < \left| \frac{\log \alpha}{\log 10} - \frac{\ell + m + n}{k} \right| < \frac{17.4}{k \cdot 10^{\ell}}. \tag{4.15}$$

Assume that $\ell > 55$. Then, it can be seen that

$$\frac{10^{\ell}}{2(17.4)} > 2.87 \times 10^{53} > 3.3 \times 10^{52} > k,$$

and so we have

$$\left| \frac{\log \alpha}{\log 10} - \frac{\ell + m + n}{k} \right| < \frac{17.4}{k \cdot 10^{\ell}} < \frac{1}{2k^2}.$$

From the known properties of continued fraction (Lemma 2.4), we conclude that the rational $\dfrac{\ell + m + n}{k}$ is a convergent to $\kappa := \dfrac{\log \alpha}{\log 10}$. So, $\dfrac{\ell + m + n}{k}$ is of the form p_t/q_t for some t. We have

$$q_{109} > 3.3 \times 10^{52} > k.$$

Thus, $t \in \{0, 1, 2, \ldots, 108\}$. By Lemma 2.4, we get

$$\frac{1}{(a(M) + 2) \cdot k^2} \leq \left| \frac{\log \alpha}{\log 10} - \frac{\ell + m + n}{k} \right| < \frac{17.4}{k \cdot 10^{\ell}}.$$

Since $a(M) = \max\{a_i : i = 0, 1, 2, \ldots, 108\} = 106$, we get

$$\ell < \frac{\log\left(17.4 \cdot (106 + 2) \cdot 3.3 \times 10^{52}\right)}{\log 10} < 55.8.$$

This contradicts the fact that $\ell > 55$. Thus $\ell \leq 55$. Therefore, the bound

$$\ell \leq 57, \tag{4.16}$$

holds in both cases. Substituting this upper bound for ℓ into (4.6) and combining the new bound obtained with (4.10), we get

$$n < 6 \times 10^{29} \cdot \log^2 n,$$

which implies $n < 1.2 \times 10^{34}$ using Lemma 2.2. Hence, by Lemma 4.1 we have $k < 2.8 \times 10^{35}$. Next, we need to reduce the bound on m. We return to (4.5) and define

$$z_5 := \log(\Lambda_5 + 1) = k \log \alpha - (n + m) \log 10 + \log\left(\frac{729}{d_1 d_2 d_3 (10^{\ell} - 1)}\right).$$

From the inequality (4.5) and $m \geq 1$, we conclude that

$$0 < \left| k \frac{\log \alpha}{\log 10} - (n + m) + \frac{\log\left(729/(d_1 d_2 d_3 (10^{\ell} - 1))\right)}{\log 10} \right| < \frac{3.5}{10^m}. \tag{4.17}$$

Now we have to study the following two cases.

Case a: $(\ell, d_1, d_2, d_3) \neq (1, 1, 9, 9), (1, 9, 9, 1), (1, 9, 1, 9)$.

We can apply Lemma 2.3 with the following data

$$\tau := \frac{\log\alpha}{\log 10}, \ \mu := \frac{\log\left(729/(d_1d_2d_3(10^\ell - 1))\right)}{\log 10}, \ A := 3.5, \ B := 10,$$

and $M := 2.8 \times 10^{35}$. Using Mathematica, we found that the denominator of the 97th convergent

$$\frac{p_{97}}{q_{97}} = \frac{3106590240739929077205211403373423170367494081}{1486494721421806760939503540391671593911615026\!0}$$

of τ exceeds $6M$. It follows that the inequality (4.17) has no solution for

$$m = w \geq \frac{\log(Aq_{97}/\varepsilon)}{\log 10} \geq \frac{\log(Aq_{97}/4.58528 \times 10^{-12})}{\log 10} \geq 58.0549.$$

Hence, we obtain $m \leq 58$.

Case b: $(\ell, d_1, d_2, d_3) \in \{(1, 1, 9, 9), (1, 9, 9, 1), (1, 9, 1, 9)\}$.

From (4.17), we can see that

$$0 < \left|\frac{\log\alpha}{\log 10} - \frac{m+n}{k}\right| < \frac{3.5}{k \cdot 10^m}. \tag{4.18}$$

Thus, for $m \geq 39$, we have

$$\frac{10^m}{2(3.5)} > 10n\log 10 > 2.8 \times 10^{35} > k,$$

and so

$$\left|\frac{\log\alpha}{\log 10} - \frac{m+n}{k}\right| < \frac{3.5}{k \cdot 10^m} < \frac{1}{2k^2}.$$

It follows that $\dfrac{m+n}{k}$ is of the form p_t/q_t for some t. Moreover, we have

$$q_{73} > 2.8 \times 10^{35} > 10n\log 10 > k.$$

Thus, $t \in \{0, 1, 2, \ldots, 72\}$. By Lemma 2.4, we get

$$\frac{1}{(a(M)+2)\cdot k^2} \leq \left|\frac{\log\alpha}{\log 10} - \frac{m+n}{k}\right| < \frac{3.5}{k \cdot 10^m}.$$

Since $a(M) = \max\{a_i : i = 0, 1, 2, \ldots, 72\} = 106$, we obtain

$$\frac{3.5}{10^{39}} \geq \frac{3.5}{10^m} > \frac{1}{108 \cdot k} > \frac{1}{108 \cdot 2.8 \times 10^{35}},$$

which is a contradiction. Therefore, $m \leq 38$. In both cases a) and b) we can consider

$$m \leq 58. \tag{4.19}$$

Using the inequalities (4.16) and (4.19) together and substituting these upper bounds into (4.10), we get

$$n < 1.2 \times 10^{16}\log n,$$

which leads to $n < 9 \times 10^{17}$ and hence to $k < 2.1 \times 10^{19}$. To reduce the bound on n we must return to inequality (4.9) and put

$$z_6 := \log(\Lambda_6 + 1) = k\log\alpha - n\log 10 + \log\left(\frac{729}{d_1d_2d_3(10^\ell - 1)(10^m - 1)}\right).$$

For $n \geq 2$, we can easily see that

$$0 < \left| k\frac{\log\alpha}{\log 10} - n + \frac{\log\left(729/\left(d_1d_2d_3(10^\ell - 1)(10^m - 1)\right)\right)}{\log 10}\right| < \frac{1.8}{10^{n-1}}. \tag{4.20}$$

It is also appropriate here to take into account the following two cases according to the values of the variables.

Case I: $729 \neq d_1d_2d_3(10^\ell - 1)(10^m - 1)$.

Therefore, we can apply Lemma 2.3 to (4.20) with $B := 10$,

$$\tau := \frac{\log\alpha}{\log 10}, \quad \mu := \frac{\log\left(729/\left(d_1d_2d_3(10^\ell - 1)(10^m - 1)\right)\right)}{\log 10}, \quad A := 1.8,$$

and $M := 2.1 \times 10^{19}$. We see that the denominator of the 44th convergent

$$\frac{p_{44}}{q_{44}} = \frac{259791952914951895804}{1243097211893507332887}$$

of τ exceeds $6M$. Thus, the inequality (4.20) has no solution for

$$n - 1 = w \geq \frac{\log(Aq_{44}/\varepsilon)}{\log 10} \geq \frac{\log(Aq_{44}/0.0000149094)}{\log 10} \geq 26.1763.$$

Hence, we obtain $n \leq 27$.

Case II: $729 = d_1d_2d_3(10^\ell - 1)(10^m - 1)$.

Using the fact that $729 = d_1d_2d_3(10^\ell - 1)(10^m - 1)$ and $g = 10$, it is easy to see from Eq. (1.4) that $L_k = 10^n - 1$. Thus, $9 \mid L_k$ so $6 \mid k$, therefore $3 \mid k$ which makes L_k even, a contradiction since $10^n - 1$ is odd. In both cases I and II, we can now consider $n \leq 27$.

In the light of the above results, we need to check Eq. (1.4) in the case $g = 10$ for $1 \leq d_1, d_2, d_3 \leq 9$, $1 \leq n \leq 27$, $1 \leq k \leq 621$, $1 \leq \ell \leq 57$ and $1 \leq m \leq 58$. A quick inspection using Maple reveals that Diophantine equation (1.4) has only the solution listed in the statement of Theorem 4.2. This ends the proof of Theorem 4.2. □

5 Concluding remarks

In this section, we bring some observations about the Diophantine equations studied in this paper. First, if $\ell = d_1 = 1$, then the Eqs. (1.3) and (1.4) become

$$F_k = a\left(\frac{g^m - 1}{g - 1}\right)\cdot b\left(\frac{g^n - 1}{g - 1}\right), \tag{5.1}$$

and

$$L_k = a\left(\frac{g^m - 1}{g - 1}\right)\cdot b\left(\frac{g^n - 1}{g - 1}\right), \tag{5.2}$$

where a, b, k, m and n are positive integers such that $1 \leq a, b \leq g - 1$ and $g \geq 2$ with $m \leq n$. Note that the two equations above were studied earlier in papers [7] and [10] where the authors exclusively mention the base g such that $2 \leq g \leq 10$. However, the method developed in this paper generalizes the results of these authors and allows us to deduce the following results which follow immediately from Theorem 1.1.

Corollary 5.1 *Let $g \geq 2$ be an integer.*

(1) The Diophantine equation (5.1) has only finitely many solutions in positive integers k, a, b, m, n such that $1 \leq a, b \leq g - 1$.
(2) The Diophantine equation (5.2) has only finitely many solutions in positive integers k, a, b, m, n such that $1 \leq a, b \leq g - 1$.

Next, when we take $g = 2$ in (1.3) and (1.4), we get the following result which gives a link between Fibonacci, Lucas and Mersenne numbers.

Corollary 5.2 *If F_k and L_k are expressible as products of three Mersenne numbers, then we have*

$$F_k \in \{1, \ 3, \ 21\} \quad \textit{and} \quad L_k \in \{1, \ 3, \ 7\}.$$

Proof The proof is similar to those of Theorems 3.2 and 4.2. □

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Declarations

Conflict of interest There is no conflict of interest related to this paper or this submission. The authors have freely chosen this journal for publication without any consideration.

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