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Diophantine pairs that induce certain Diophantine triples

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ABSTRACT

Diophantine tuples are sets of positive integers with the property that the product of any two elements in the set increased by the unity is a square. In the main theorem of this paper it is shown that any Diophantine triple, the second largest element of which is between the square and four times the square of the smallest one, is uniquely extended to a Diophantine quadruple by joining an element exceeding the largest element in the triple. A similar result is obtained under the hypothesis that the two smallest elements have the form T^2+2T, $4T^4+8T^3-4T$ for some positive integer T, which we encounter as an exceptional case. The main theorem implies that the same is valid for triples with smallest elements KA^2, $4KA^4 \pm 4A$ for some positive integers A and $K \in \{1, 2, 3, 4\}$.
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1. Introduction

This paper is devoted to Diophantine tuples, defined as sets of positive integers with the property that the product of any two elements in the set increased by the unity is a square. When the set has cardinality 2 (3, 4 or 5), we shall speak of a Diophantine pair (triple, quadruple or quintuple, respectively).

Since Euler it is known that each Diophantine pair $\{a, b\}$ can be extended to a Diophantine triple by taking $c = a + b + 2r$, where $r = \sqrt{ab + 1}$. Arkin, Hoggatt and Strauss [2] and independently Gibbs [21] found a possibility to produce Diophantine quadruples: they associated to an arbitrary Diophantine triple $\{a, b, c\}$ the integer

$$d_+ = a + b + c + 2abc + 2\sqrt{(ab + 1)(bc + 1)(ca + 1)} \tag{1}$$

and showed that $\{a, b, c, d_+\}$ is a Diophantine quadruple, usually referred to as regular. Moreover, they conjectured that this is the only possibility to extend a Diophantine triple $\{a, b, c\}$ to a quadruple by an integer greater than $\max\{a, b, c\}$.

Since there are no Diophantine quintuples, as shown quite recently in [24] and (for an even more general case) in [4], the main open question regarding Diophantine tuples is the unique extendability of Diophantine triples. This has been established under various additional conditions. From results available in the literature we quote here only those needed in subsequent proofs. The willing reader can find references to previous / other relevant results on the web page http://web.math.pmf.unizg.hr/~duje/dtuples.html.

Let $\{a, b, c, d\}$ be a Diophantine quadruple with $a < b < c < d$. The available results can be classified according to several criteria. On the one hand, there are those in which a and b are supposed to be 'close to each other'. Thus, from [15] and [8] it is known that in order to have $d > d_+$ it is necessary to have $b > 4000$. In the same category enter the unicity of extension to a Diophantine quadruple for any pair $\{a, b\}$ of the type $\{k, k + 2\}$ $(k \geq 1)$ [6,18] or for which $b < a + 4\sqrt{a}$ (see [16]). On the other hand, there are articles dealing with the problem when c is 'much bigger' than b. Thus, Miyazaki and the third author confirmed in [20] the validity of the conjecture when $c \geq 721.8b^4$. This result is slightly improved in [11], where it is shown that it holds assuming that $c \geq 200b^4$.

Numerous papers deal with parametric Diophantine tuples. In [10], the uniqueness of extension has been established for all members of the two-parameter families $\{k, A^2k + 2\varepsilon A, (A+1)^2k+2\varepsilon(A+1)\}$ with positive integers k and A and $\varepsilon \in \{\pm 1\}$. The first infinite two-parameter family of Diophantine triples with unique extension is pointed out in [9]. As we shall shortly see, this family is relevant for the work reported here.

The present article draws attention to a different context in which the conjecture is valid. Our main result asserts that if $\{a, b, c, d\}$ is a Diophantine quadruple with b, roughly speaking, between a^2 and $4a^2$, then $d = d_+$. This is for the first time in the literature when b compares not to a but to a^2.

Theorem 1.1. *Let $\{a, b\}$ be a Diophantine pair satisfying*

$$a\left(a+\frac{7}{2}-\frac{1}{2}\sqrt{4a+13}\right) \leq b \leq 4a^2+a+2\sqrt{a}. \tag{2}$$

If $\{a, b, c, d\}$ is a Diophantine quadruple with $b < c < d$, then $d = d_+$.

Such a result opens new prospects for the study of parametric Diophantine pairs. Theorem 1.1 immediately implies the following (see Section 2).

Theorem 1.2. *Let a, b be positive integers defined by $a = KA^2$, $b = 4KA^4 + 4\varepsilon A$ with A a positive integer, $K \in \{1, 2, 3, 4\}$ and $\varepsilon \in \{\pm 1\}$. If $\{a, b, c, d\}$ is a Diophantine quadruple with $b < c < d$, then $d = d_+$.*

Note that in case c is taken canonically (viz., $c = c_\nu^\tau$ with c_ν^τ as defined in Section 3), it has been already known in [9, Theorem 1.1] that the assertion of Theorem 1.2 is valid for arbitrary positive integer K. Therefore, Theorem 1.2 can be regarded as a generalization of [9, Theorem 1.1] in the cases where $K \in \{1, 2, 3, 4\}$. Furthermore, it also generalizes [16, Theorem 1.8] and [22, Theorem 3], as seen from the fact that a specialization $A = 1$ of the pair in Theorem 1.2 gives $\{a, b\} = \{K, 4K + 4\varepsilon\}$.

Section 3 contains a crucial observation for our results, providing an explanation for the relevance of the hypothesis (2) in Theorem 1.1. The proof of the main result Theorem 1.1 combines a thorough study of systems of Pellian equations with extensive computations. Since the details of the reasoning developed in Section 4 varies according to the gap between b and c, we have to split the argument into several pieces distributed over three subsections. In the final section, the extendibility of a parametric family appearing as the exceptional cases for condition (2) is investigated in the case where $c \neq c_\nu^\tau$ for any ν and τ. In the situation where c can be expressed as $c = \gamma_1^+$ and $c = \gamma_2^+$ with some recurrence sequence $\{\gamma_\nu^+\}$ introduced in Section 5, we need improvements of the procedure encountered in the literature for finding better lower bounds for solutions. Another twist of the usual method appears in case $c = \gamma_1^+$, where we obtain a suitable upper bound for the solution by applying Matveev's theorem in 1998 ([27, Theorem 2.1]), which requires to check the "Kummer condition" (see Lemma 5.3). Since the case where (2) is satisfied and $c = c_\nu^\tau$ for some ν and τ is already examined in the preceding section, the proof of the main result is complete as soon as the following is established.

Theorem 1.3. *Let T be a positive integer and let $a = T^2 + 2T$, $b = 4T^4 + 8T^3 - 4T$. If $\{a, b, c, d\}$ is a Diophantine quadruple with $a < b < c < d$, then $d = d_+$.*

2. Proof of Theorem 1.2

Since the case where $A = 1$, that is, $\{a, b\} = \{K, 4K + 4\varepsilon\}$, is already shown in [16, Theorem 1.8] and [22, Theorem 3], we may assume that $A \geq 2$. Then, it is easy to check that inequalities (2) hold for each $K \in \{1, 2, 3, 4\}$. Therefore, Theorem 1.2 follows from Theorem 1.1.

3. A key lemma

Let a, b, r with $a < r < b$ be positive integers such that $ab + 1 = r^2$. Following [9], we define an integer $c = c_\nu^\tau$ by

$$c_\nu^\tau = \frac{1}{4ab}\left\{(\sqrt{b}+\tau\sqrt{a})^2(r+\sqrt{ab})^{2\nu} + (\sqrt{b}-\tau\sqrt{a})^2(r-\sqrt{ab})^{2\nu} - 2(a+b)\right\}$$

with ν a positive integer and $\tau \in \{\pm\}$. For reader's convenience, we write down those terms of these sequences we shall use in subsequent proofs:

$$\begin{aligned}
c_1^\tau &= a + b + 2\tau r,\\
c_2^\tau &= 4ab(a + b + 2\tau r) + 4(a + b + \tau r),\\
c_3^\tau &= 16a^2b^2(a + b + 2\tau r) + 8ab(3a + 3b + 4\tau r) + 3(3a + 3b + 2\tau r).
\end{aligned}$$

The crucial observation on which the present work relies is formalised in the next result.

Lemma 3.1. *Let $\{a, b, c\}$ be a Diophantine triple with $4000 < b < c$. If*

$$a\left(a + \frac{7}{2} - \frac{1}{2}\sqrt{4a + 13}\right) \le b \le 2.98\, a^2, \tag{3}$$

then $c = c_\nu^\tau$ for some positive integer ν and some $\tau \in \{\pm\}$. In addition, if the pair $\{a, b\}$ cannot be expressed as $\{T^2 + 2T, 4T^4 + 8T^3 - 4T\}$ for any integer $T > 1$, and if inequalities (2) hold, then $c = c_\nu^\tau$ for some integer $\nu \ge 1$ and some $\tau \in \{\pm\}$.

Proof. Suppose that $c \ne c_\nu^\tau$. The proof of Lemma 4.1 in [15] allows us to assume that $0 < c' < b$, where $c' = ((s')^2 - 1)/a$ and $s' = rs - at$. Putting $b' = ((r')^2 - 1)/a$ with $r' = s'r - at'$ and $t' = rt - bs$ shows that

$$b = a + b' + c' + 2ab'c' + 2r's'T, \tag{4}$$

where $T = \sqrt{b'c' + 1}$. If $b' = 0$, then $b = a + c' + 2s'$, which in turn implies $c = c_2^-$. Hence, b' is a positive integer and $\{a, B, C, b\}$ is a regular Diophantine quadruple, where $B = \min\{b', c'\}$ and $C = \max\{b', c'\}$.

If $a \le BC$ then one can deduce from (4) that

$$b > 4aBC + a + B + C > 4a^2 + a + 2\sqrt{a}. \tag{5}$$

Below we suppose that $a > BC$. If $a \neq B + C + 2T$ then $a > 4BC + B + C$. From $a - 1 \geq 4BC + 2\sqrt{BC + 1}$ we deduce $8BC \leq 2a - 1 - \sqrt{4a + 13}$, which together with (4) entails

$$b < 4aBC + 4a \leq a\left(a + \frac{7}{2} - \frac{1}{2}\sqrt{4a + 13}\right), \tag{6}$$

in contradiction with the hypothesis.

Assume that $a = B + C + 2T$. Then, $BC < B + C + 2T$ implies either $B = 1$ or

$$C \leq \frac{B^2 + 1 + 2\sqrt{B^3 - B + 1}}{(B - 1)^2}. \tag{7}$$

In the latter case, as the right-hand side of (7) is a decreasing function of B, one can easily see that $B \leq 3$. Since $B \geq 2$, the only possible triples $\{B, C, a\}$ are

$$\{2, 4, 12\}, \quad \{3, 5, 16\},$$

which respectively induce the following Diophantine pairs $\{a, b\}$:

$$\{12, 420\}, \quad \{16, 1008\},$$

any of which does not satisfy the assumption $b > 4000$.

It remains to consider the case where $a = C + 1 + 2\sqrt{C + 1}$. Now one has $C = T^2 - 1$ and $a = T^2 + 2T$ for some integer $T > 1$. Since $\{1, C, a, b\}$ is a regular Diophantine quadruple, one also has $b = 4T^4 + 8T^3 - 4T$, which together with $4000 < b \leq 2.98a^2$ shows $T = 6$. Thus, $b/a^2 = 6888/48^2 > 2.989$, a contradiction.

Therefore, if $c \neq c_\nu^\pm$, then either $b < a^2 + (7 - \sqrt{4a + 13})a/2$ or $b > 2.98\,a^2$. The first assertion of the lemma is exactly the contraposition of this statement. The second assertion follows from (5), (6) and the last paragraph. $\square$

4. Proof of Theorem 1.1

Assume that $\{a, b, c, d\}$ is a Diophantine quadruple with $a < b < c < d_+ < d$. Let x, y, z be positive integers such that $ad + 1 = x^2$, $bd + 1 = y^2$, $cd + 1 = z^2$. Then eliminating d from these equations yields a system of Pellian equations

$$ay^2 - bx^2 = a - b, \tag{8}$$

$$az^2 - cx^2 = a - c, \tag{9}$$

$$bz^2 - cy^2 = b - c. \tag{10}$$

We know by [12, Lemma 1] that there exist a nonnegative integer l and a solution (y_2, x_2) to (8) with

$$1 \leq x_2 < \sqrt{b}, \quad 1 \leq |y_2| < \sqrt{b\sqrt{b}} \tag{11}$$

such that $x = V_l$ and $y = u'_l$, where

$$V_0 = x_2, \quad V_1 = rx_2 + ay_2, \quad V_{l+2} = 2rV_{l+1} - V_l,$$
$$u'_0 = y_2, \quad u'_1 = ry_2 + bx_2, \quad u'_{l+2} = 2ru'_{l+1} - u'_l.$$

Hence,

$$u'_l \equiv \begin{cases} y_2 \pmod{2b} & \text{if } l \text{ is even;} \\ ry_2 + bx_2 \pmod{2b} & \text{if } l \text{ is odd.} \end{cases} \tag{12}$$

After these general considerations, we can proceed with the promised proof. We examine the case where $\{a, b\} = \{T^2 + 2T, 4T^4 + 8T^3 - 4T\}$ for some integer $T > 1$ in Section 5. By Lemma 3.1, in the next three subsections we may assume that $c = c_\nu^\tau$ for some ν and τ.

4.1. The case $c = c_1^\tau$

If $c = c_1^\tau$, then $\tau = +$, that is, $c = c_1^+ = a + b + 2r$. From [20, Theorem 1.3] and [25, Theorem 8] one sees that any solutions to (9), (10) can be expressed respectively as

$$z\sqrt{a} + x\sqrt{c} = (\lambda_1\sqrt{a} + \sqrt{c})(s + \sqrt{ac})^m,$$
$$z\sqrt{b} + y\sqrt{c} = (\lambda_1\sqrt{b} + \sqrt{c})(t + \sqrt{bc})^n$$

for some nonnegative even integers m, n and $\lambda_1 \in \{\pm 1\}$, which enables us to write $x = W_m$ and $y = u_n$, where

$$W_0 = 1, \quad W_1 = s + \lambda_1 a, \quad W_{m+2} = 2sW_{m+1} - W_m,$$
$$u_0 = 1, \quad u_1 = t + \lambda_1 b, \quad u_{n+2} = 2tu_{n+1} - u_n. \tag{13}$$

Since we have $y = u_n \equiv 1 \pmod{2b}$ by (13) with n even, if l is even, then $y_2 \equiv 1 \pmod{2b}$, which together with (11) implies $y_2 = 1$ and $x_2 = 1$. If l is odd, then $ry_2 + bx_2 \equiv 1 \pmod{2b}$. Since $0 < bx_2 - r|y_2| \leq b - r < b$, one has $bx_2 - r|y_2| = 1$. However, (11) shows that $bx_2 + r|y_2| < 2bx_2 < 2b\sqrt{b}$, while

$$bx_2 + r|y_2| = (bx_2 + r|y_2|)(bx_2 - r|y_2|) = b^2 - ab - y_2^2 > 61b\sqrt{b},$$

which is a contradiction. Therefore, one obtains the following.

Lemma 4.1. *Assume that $\{a, b, c_1^+, d\}$ is a Diophantine quadruple with $d > d_+$. If $x = V_l = W_m$ has a solution, then $l \equiv m \equiv 0 \pmod{2}$ and $x_2 = y_2 = 1$.*

Now let

$$\alpha = s + \sqrt{ac}, \quad \beta = r + \sqrt{ab}, \quad \chi = \frac{\sqrt{bc} + \sqrt{ac}}{\sqrt{bc} \pm \sqrt{ab}}.$$

Since we are assuming $d > d_+$, it is easy to see that $m \geq 3$ (see the proof of [23, Lemma 1] for a similar argument). Then, the standard argument shows that

$$0 < \Lambda < \alpha^{1-2m} \tag{14}$$

where $\Lambda = l \log \beta - m \log \alpha + \log \chi$ (cf. [23, Lemma 2]). Rewriting Λ as $\Lambda = \log(\beta^\nu \chi) - m \log(\alpha/\beta)$, where $\nu = l - m$, one has $\nu \geq 2$ in a similar way to the proof of [23, Lemma 1], and from (14) with $\log \chi > \alpha^{1-2m}$ one sees that $m \log(\alpha/\beta) > \nu \log \beta$. Since $\log(\alpha/\beta) < \sqrt{a/b}$ by [10, Lemma 3.6], one can deduce the lower bound for the solution:

$$m > \nu \sqrt{\frac{b}{a}} \log \beta. \tag{15}$$

In order to get an upper bound for m, we appeal to Laurent's theorem on linear forms in two logarithms.

Proposition 4.2. ([26, Theorem 2]) *Let α_1 and α_2 be multiplicatively independent algebraic numbers with $|\alpha_1| \geq 1$ and $|\alpha_2| \geq 1$. Let b_1 and b_2 be positive integers. Consider the linear form in two logarithms*

$$\Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1,$$

where $\log \alpha_1, \log \alpha_2$ are any determinations of the logarithms of α_1, α_2, respectively. Let ρ and μ be real numbers with $\rho > 1$ and $1/3 \leq \mu \leq 1$. Set

$$\sigma = \frac{1 + 2\mu - \mu^2}{2}, \quad \lambda = \sigma \log \rho.$$

Let a_1, a_2 be real numbers such that

$$\begin{aligned} &a_i \geq \max\{\,1,\ \rho\,|\log \alpha_i| - \log|\alpha_i| + 2D\,\mathrm{h}(\alpha_i)\,\} \quad (i = 1, 2),\\ &a_1 a_2 \geq \lambda^2, \end{aligned}$$

where $D = [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}] \,/\, [\mathbb{R}(\alpha_1, \alpha_2) : \mathbb{R}]$. Let h be a real number such that

$$h \geq \max\left\{ D\left(\log\left(\frac{b_1}{a_2} + \frac{b_2}{a_1}\right) + \log \lambda + 1.75\right) + 0.06,\ \lambda,\ \frac{D \log 2}{2} \right\}.$$

Then we have

$$\log|\Lambda| \geq -C\left(h+\frac{\lambda}{\sigma}\right)^2 a_1a_2 - \sqrt{\omega\theta}\left(h+\frac{\lambda}{\sigma}\right) - \log\left(C'\left(h+\frac{\lambda}{\sigma}\right)^2 a_1a_2\right),$$

where

$$\begin{aligned}
\sigma &= \frac{1+2\mu-\mu^2}{2}, \quad \lambda = \sigma\log\rho,\\
\omega &= 2\left(1+\sqrt{1+\frac{1}{4H^2}}\right), \quad \theta = \sqrt{1+\frac{1}{4H^2}}+\frac{1}{2H},\\
h &\geq \max\left\{4\left(\log\left(\frac{b_1}{a_2}+\frac{b_2}{a_1}\right)+\log\lambda+1.75\right)+0.06,\ \lambda,\ 2\log 2\right\},\\
H &= \frac{h}{\lambda}+\frac{1}{\sigma},\\
C &= \frac{\mu}{\lambda^3\sigma}\left(\frac{\omega}{6}+\frac{1}{2}\sqrt{\frac{\omega^2}{9}+\frac{8\lambda\omega^{5/4}\theta^{1/4}}{3\sqrt{a_1a_2}H^{1/2}}+\frac{4}{3}\left(\frac{1}{a_1}+\frac{1}{a_2}\right)\frac{\lambda\omega}{H}}\right)^2,\\
C' &= \sqrt{\frac{C\sigma\omega\theta}{\lambda^3\mu}}.
\end{aligned}$$

Suppose first that (3) holds. We apply Proposition 4.2 with $b_1 = m$, $b_2 = 1$, $\alpha_1 = \alpha/\beta$ and $\alpha_2 = \beta^\nu\chi$. Note that (3) together with $b > 4000$ implies that

$$\sqrt{\frac{b}{2.98}} < a < \sqrt{\frac{b}{0.923}}. \tag{16}$$

As seen in the proof of [10, Proposition 6.1], one has

$$h(\alpha_1) = \frac{1}{2}\log\alpha, \quad h(\chi) \leq \frac{1}{4}\log(bc^2(c-a)),$$

which shows that it is sufficient to have

$$\begin{aligned}
a_1 &\geq 4\log\alpha + (\rho-1)(0.923b)^{-1/4},\\
a_2 &\geq \nu(\rho+3)\log\beta + 2\log(bc^2(c-a)) + (\rho-1)\log\chi.
\end{aligned}$$

A similar argument to [10, Lemmas 3.7, 3.9] implies that $\beta > 1.999r$ and

$$bc^2(c-a) < \left(1+2\sqrt{\frac{a}{b}+\frac{1}{b^2}}\right)\left(\frac{1}{r}+\frac{1}{a}\right)^4 r^8 < 10^{-8}\beta^8.$$

We now fix $\rho = 37$, $\mu = 0.63$. Since

$$\chi \leq \frac{\sqrt{bc}+\sqrt{ac}}{\sqrt{bc}-\sqrt{ab}} < 1 + \frac{2}{\sqrt{b/a}-\sqrt{b/c}},$$

we obtain

$$a_1 \geq f_1(a,b)\log\alpha, \quad a_2 \geq (40\nu + f_2(a,b))\log\beta,$$

where

$$f_1(a,b) = 4 + \frac{36}{0.923^{1/4}b^{1/4}\log\alpha},$$
$$f_2(a,b) = 16\left(1 - \frac{\log(10)}{\log\beta}\right) + \frac{36\log\left(1 + \frac{2}{\sqrt{b/a}-\sqrt{b/c}}\right)}{\log\beta} < 16.$$

Furthermore, since $\alpha > \beta$ implies that

$$\frac{b_1}{a_2} + \frac{b_2}{a_1} \leq \frac{m}{(40\nu+16)\log\beta} + \frac{1}{f_1(a,b)\log\alpha}$$
$$< \frac{m + (40\nu+16)/f_1(a,b)}{(40\nu+16)\log\beta},$$

one can deduce that

$$h = 4\log\left(\frac{m + (40\nu+16)/f_1(a,b)}{(40\nu+16)\log\beta}\right) + 11.913.$$

Suppose for the moment that $b > 10^{14}$. Then, in view of inequalities (16), we may take

$$a_1 = 4.001\log\alpha, \quad a_2 = (40\nu+16)\log\beta, \tag{17}$$
$$h = 4\log\left(\frac{m + 9.998\nu + 4}{(40\nu+16)\log\beta}\right) + 11.913.$$

If $h \leq 28.74$, then inequality (15) shows that

$$\nu\sqrt{\frac{b}{a}}\log\beta < m < (40\nu+16)\exp(4.20675)\log\beta,$$

which together with $\nu \geq 2$ yields $b/a < 1.039 \cdot 10^7$. If $h > 28.74$, then Proposition 4.2 together with the estimate $0 < \Lambda < \alpha^{1-2m}$ implies that

$$\frac{2m-1}{4.001(40\nu+16)\log\beta} < C\left(h + \frac{\lambda}{\sigma}\right)^2 + \frac{\sqrt{\omega\theta}\left(h+\frac{\lambda}{\sigma}\right) + \log\left(C'\left(h+\frac{\lambda}{\sigma}\right)^2 a_1a_2\right)}{a_1a_2},$$

which yields

$$\frac{m}{(40\nu+16)\log\beta} < 67.26.$$

It follows from inequality (15) that $b/a < 1.04231 \cdot 10^7$.

Therefore, it remains to obtain a contradiction for $4000 < b < 1.0868 \cdot 10^{14}$, $37 \le a < 1.043 \cdot 10^7$ and $385 \le r < 3.367 \cdot 10^{10}$, which can be done by using the reduction method introduced in [3].

Because it is the most difficult case in all the reduction procedures used in this paper (in terms of running time), we will give some details here. We were solving the equation $z = v_{2m} = w_{2n}$ for $z_0 = z_1 = \pm 1$. In order to bound from above the index m we have used the linear form in logarithms from [13, relation (60)]. Using the rough estimates given there in conjunction with [28], we got the initial estimate $m < M = 10^{21}$ in all cases. Now, for every a we find all r which satisfy $r^2 \equiv 1 \pmod a$ and we have lower and upper bounds for r, depending on a, from the bounds for $b = (r^2-1)/a$. Depending on which upper bound is smaller, we take $b < 2.98a^2$ or $b < 1.04231 \cdot 10^7 a$. We also check if $r \equiv \pm 1 \pmod a$. If that is the case, we do not have to do the reduction because of the results proved in [10]. Now for fixed a and r we find b and $c = a+b+2r$ and do the variant of the reduction process described in [14, Lemma 5]. We have done it in PARI-GP [29], running 20 programs simultaneously (we have used the command `forstep` with different starting a), with real precision of 77 significant digits, on Intel® Core™ i7-7700 CPU@3.60–4.20 GHz.

It took 55 days to finish all of them. After at most 2 steps of reduction we got $m < 2$ in all cases, which means that the only solution is $z = v_2 = w_2$ that gives us the already known extension to a quadruple with $d = d_+$.

Suppose second that $2.98a^2 < b \le 4a^2 + a + 2\sqrt{a}$. From $b > 4000$ it then follows $a \ge 32$. Note that we are still in the case where $c = c_1^+ = a + b + 2r$.

Assume now that $b > 10^{13}$. Then, $a < \sqrt{b/2.98}$ and

$$a \ge \sqrt{\frac{b}{4 + 1/a + 2/(a\sqrt{a})}} > \sqrt{\frac{b}{4.001}}.$$

In applying Proposition 4.2, since $\log \alpha/\beta < \sqrt{a/b} < (2.98b)^{-1/4}$ and $bc^2(c-a) < 10^{-8}\beta^8$ can be seen in the same way as in case (3), it is sufficient that

$$a_1 \ge \left(4 + \frac{36}{2.98^{1/4} b^{1/4} \log \alpha}\right) \log \alpha,$$
$$a_2 \ge (40\nu + 16) \log \beta.$$

Thus, one may take a_1, a_2, h given by (17), exactly the same as in case (3). It follows from Proposition 4.2 and inequality (15) that $b/a < 1.04231 \cdot 10^7$, $4000 < b < 3.6457 \cdot 10^{13}$ and $32 \le a < 3.498 \cdot 10^6$. It only remains to perform reduction for the triples satisfying these inequalities and $2.98a^2 < b \le 4a^2 + a + 2\sqrt{a}$.

We have done this proceeding as in the case where inequality (3) holds. With the same processor, it took 28 days to reach the conclusion that the only extension of the triple $\{a, b, a + b + 2r\}$ is with $d = d_+$.

4.2. The case $c = c_2^\tau$

Note that c is even, so both s and t are odd. By [20, Theorem 1.3] and [11, Lemma 2.3], we may assume that $v_m = w_n$ holds, where either of the following holds:

(A) Both m and n are even with $z_0 = z_1 = \varepsilon_1 \in \{\pm 1\}$;
(B) Both m and n are odd with $z_0 = \varepsilon_2 t$, $z_1 = \varepsilon_2 s$ and $\varepsilon_2 \in \{\pm 1\}$.

In the case of (B), any solution to (10) can be written as $y = u_n$, where

$$u_0 = r, \quad u_1 = rt + \varepsilon_2 bs, \quad u_{n+2} = 2tu_{n+1} - u_n. \tag{18}$$

Since $t = t_2^\tau = 2rb + \tau(2ab+1) \equiv \tau \pmod{2b}$, one has

$$u_n \equiv \pm(b - r) \pmod{2b}, \tag{19}$$

since n is odd. Comparing (19) with (12), one sees that

$$b - r \equiv \begin{cases} \pm y_2 \pmod{2b} & \text{if } l \text{ is even;} \\ \pm(bx_2 - r|y_2|) \pmod{2b} & \text{if } l \text{ is odd.} \end{cases}$$

If l is even, then $b \equiv r \pm y_2 \pmod{2b}$. However, (11) shows that $0 < |r \pm y_2| \le r + |y_2| < r + \sqrt{b\sqrt{b}} < b$, which is a contradiction. If l is odd, then $b - r \equiv \pm(bx_2 - r|y_2|) \pmod{2b}$. Since both sides have absolute value between 0 and b, one has

$$b - r = bx_2 - r|y_2|.$$

Combining this with $ay_2^2 - bx_2^2 = a - b$ yields

$$x_2^2 + 2a(b - r)x_2 - 2ab - 1 + 2ar = 0.$$

By $x_2 > 0$, one obtains $x_2 = 1$, and hence $y_2 = \pm 1$. Moreover, since $t \equiv \tau \pmod{b}$, one has $u_n \equiv \tau r \pmod{b}$ and $u_l' \equiv y_2 r \pmod{b}$, which together show that $y_2 r \equiv \tau r \pmod{b}$. Since $\gcd(b, r) = 1$, one obtains $y_2 = \tau$.

Since the case where both m and n are even can be treated exactly in the same way as in the proof of Lemma 4.1, the following holds.

Lemma 4.3. *Assume that $\{a, b, c_2^\tau, d\}$ is a Diophantine quadruple with $d > d_+$. Assume that $z = v_m = w_n$ has a solution.*

(i) *If both m and n are even, then l is even and $(x_2, y_2) = (1, 1)$.*
(ii) *If both m and n are odd, then l is odd and $(x_2, y_2) = (1, \tau)$.*

Our next concern is to obtain lower bounds for m in terms of a and b. First consider case (A).

Lemma 4.4. *Assume that $v_m = w_n$ holds with $m \equiv n \equiv 0 \pmod 2$. Put $\Delta = m - l/2$ for $\tau = -$ and $\Delta = l/2 - m$ for $\tau = +$. If $m > 0$, then $\Delta \geq 1$.*

Proof. Assume first that $\tau = -$. We show that $V_{2m} > W_m$ by induction. One has $V_2 = 2a(b + r) + 1 > s + a \geq W_1$. For $m \geq 2$, the inductive assumption together with $s = 2ab - 2ar + 1 < 2r^2 - 2$ shows that

$$\begin{aligned} W_m &< 2sW_{m-1} < (4r^2 - 4)V_{2m-2} = 2r(V_{2m-1} + V_{2m-3}) - 4V_{2m-2} \\ &= V_{2m} + V_{2m-4} - 2V_{2m-2} < V_{2m}. \end{aligned}$$

Assume second that $\tau = +$. Then, $V_2 = 2a(b + r) + 1 = s$ and $W_1 = s \pm a$, which mean that $V_l = W_m$ with $m > 0$ has a solution only for $l > 2$. Thus, it suffices to show $V_{2m} < W_m$ for $m \geq 2$. Since $c > 4ab(b + 2r)$ and

$$\begin{aligned} V_4 &= 8a^2b(b + r) + 8ab + 4ar + 1, \\ W_2 &= 2ac \pm 2as + 1 > 8a^2b(b + 2r) - 2a(2ab + 2ar + 1) + 1, \end{aligned}$$

one has $V_4 < W_2$. For $m \geq 3$, the inductive assumption together with $2s - 1 = 4ab + 4ar + 1 > 4r^2$ implies that

$$W_m > (2s - 1)W_{m-1} > 4r^2V_{2m-2} > V_{2m}. \quad \square$$

Lemma 4.5. *Assume that $v_m = w_n$ holds with $m \equiv n \equiv 0 \pmod 2$.*

(1) *If $\tau = -$ and $b > 10^{13}$, then $m > \frac{2\Delta - 1}{1.0013}\sqrt{\frac{b}{a}}\log\beta$, where $\Delta = m - l/2$.*
(2) *If $\tau = +$ and $b > 10^{13}$, then $m > \frac{2\Delta}{1.0003}\sqrt{\frac{b}{a}}\log\beta$, where $\Delta = l/2 - m$.*

Proof. Note that in view of Lemma 4.4, the right-hand side of each inequality in (1) and (2) is positive. Observe that

$$\begin{aligned} \beta^2 - \alpha &= \sqrt{a}(2r\sqrt{b} - \sqrt{c} - 2\tau r\sqrt{a}), \\ 2r\sqrt{b} - \sqrt{c} &= \frac{4b(ab + 1) - c}{2r\sqrt{b} + \sqrt{c}} = -\frac{4r(ar + 2\tau ab + \tau)}{2r\sqrt{b} + \sqrt{c}} \end{aligned}$$

and

$$\frac{\beta^2 - \alpha}{2r\sqrt{a}} = -\frac{\tau\sqrt{ac} + 2\tau r\sqrt{ab} + 4\tau ab + 2ar + 2\tau}{\sqrt{c} + 2r\sqrt{b}}. \tag{20}$$

Assume that $b > 10^{13}$. Inequalities (2) imply that

$$a > 1.581 \cdot 10^6 \quad \text{and} \quad b > 0.9992a^2.$$

(1) In the case of $\tau = -$, one has

$$a + b - 2r = \frac{(b-a)^2 - 4}{a + b + 2r} > 0.998b,$$

which shows that

$$c = c_2^- = 4ab(a + b - 2r) + 4(a + b - r) > 3.992ab^2.$$

It follows from (20) that

$$\frac{\beta^2 - \alpha}{2r\sqrt{a}} < \frac{\sqrt{3.992}ab + 6ab}{\sqrt{3.992}ab + 2r\sqrt{b}} < 2.00051\sqrt{a},$$

which yields $\beta^2 - \alpha < 4.0011ar$. Since $\alpha > 2\sqrt{ac} > 3.99599ab$, one has

$$\frac{\beta^2 - \alpha}{\alpha} < \frac{4.0011r}{3.99599b} < 1.0013\sqrt{\frac{a}{b}}.$$

Now it is easy to see from (14) that $(l+1)\log\beta > m\log\alpha$. Therefore, putting $\Delta = m - l/2$ leads us to

$$\begin{aligned} \frac{2\Delta - 1}{m} &= 2 - \frac{l+1}{m} < 2 - \frac{\log\alpha}{\log\beta} \\ &= \frac{\log(\beta^2/\alpha)}{\log\beta} < \frac{\beta^2 - \alpha}{\alpha\log\beta} < \frac{1.0013}{\sqrt{b/a}\log\beta}, \end{aligned}$$

from which the desired inequality is derived.

(2) In the case of $\tau = +$, since $c = c_2^+ = 4ab(a + b + 2r) + 4(a + b + r) > 4ab^2$ and

$$2r\sqrt{ab} + 4ab + 2ar + 2 < 6.0016ab,$$

one sees from (20) that

$$\frac{\alpha - \beta^2}{2r\sqrt{a}} < \frac{2ab + 6.0016ab}{2\sqrt{a}b + 2\sqrt{a}b} = 2.0004\sqrt{a},$$

and hence $\alpha - \beta^2 < 4.0008ar$. Combining this inequality with $\beta^2 > 4ab$, one has

$$\frac{\alpha - \beta^2}{\beta^2} < \frac{4.0008ar}{4ab} < 1.0003\sqrt{\frac{a}{b}}.$$

In view of the inequality $l \log \beta < m \log \alpha$ obtained from (14), it follows from $\varDelta = l/2 - m$ that

$$\frac{2\varDelta}{m} = \frac{l}{m} - 2 < \frac{\log \alpha}{\log \beta} - 2 < \frac{\alpha - \beta^2}{\beta^2 \log \beta} < \frac{1.0003}{\sqrt{b/a}\log \beta},$$

which gives the inequality as asserted. $\square$

Proposition 4.6. *Let $\{a, b, c, d\}$ be a Diophantine quadruple with $a < b < c = c_2^\tau < d_+ < d$ with $\tau \in \{\pm\}$. Assume that $z = v_m = w_n$ for some even m and n with $z_0 = z_1 \in \{\pm 1\}$.*

(1) *If $\tau = -$, then $b/a < 7.035 \cdot 10^6$ and $b < 4.951 \cdot 10^{13}$.*
(2) *If $\tau = +$, then $b/a < 7.018 \cdot 10^6$ and $b < 4.929 \cdot 10^{13}$.*

Proof. (1) In the case of $\tau = -$, rewrite the linear form $\varLambda$ appearing in (14) as

$$\varLambda = \frac{l}{2}\log\left(\frac{\beta^2}{\alpha}\right) - \log\left(\frac{\alpha^{\varDelta}}{\chi}\right),$$

and assume that $b > 10^{13}$. We apply Proposition 4.2 with $b_1 = 1$, $b_2 = l/2$, $\alpha_1 = \alpha^{\varDelta}/\chi$, $\alpha_2 = \beta^2/\alpha$. Since the conjugates of α_2 greater than 1 are

$$\frac{(r+\sqrt{ab})^2}{s+\sqrt{ac}} \quad \text{and} \quad \frac{(r+\sqrt{ab})^2}{s-\sqrt{ac}},$$

one has

$$h(\alpha_2) = \frac{1}{4}\log(r+\sqrt{ab})^4 = \log \beta.$$

As in [10, Proof of Proposition 6.1], one has

$$h(\chi) \le \frac{1}{4}\log\left(bc(c-a)(\sqrt{a}+\sqrt{b})^2\right). \tag{21}$$

Since $3.992ab^2 < c < 4ab^2$, $b(\sqrt{a}+\sqrt{b})^2 < 1.0012b^2 < c/(3.987a)$ and $b < 4.001a^2$ by (2), one gets

$$\begin{aligned} h(\chi) &< \frac{1}{4}\log\left(\frac{c^3}{3.987a}\right) < \frac{1}{4}\log\left(\frac{64}{3.987}a^2b^6\right) \\ &< \frac{1}{4}\log\left(\frac{64}{3.987}(4.001)^{4/3}\left(\frac{\alpha}{3.9959}\right)^{14/3}\right) < \frac{7}{6}\log \alpha. \end{aligned}$$

It follows that

$$h(\alpha_1) = h(\alpha^{\varDelta}/\chi) \le \varDelta h(\alpha) + h(\chi) < \frac{3\varDelta + 7}{6}\log \alpha.$$

Hence, a_1 and a_2 should satisfy

$$\begin{aligned} a_1 &\geq (\rho-1)\log\left(\frac{\alpha^{\Delta}}{\chi}\right) + 8\cdot\frac{3\Delta+7}{6}\log\alpha \\ &= \left(\Delta(\rho+3)+\frac{28}{3}\right)\log\alpha - (\rho-1)\log\chi, \\ a_2 &\geq (\rho-1)\log\left(\frac{\beta^2}{\alpha}\right) + 8\log\beta = 4\log\alpha + (\rho+3)\log\left(\frac{\beta^2}{\alpha}\right). \end{aligned}$$

Now fix $\rho = 37$ and $\mu = 0.63$. Since

$$\frac{\beta^2}{\alpha} < \frac{(r+\sqrt{ab})^2}{2\sqrt{ac}} < \frac{(r+\sqrt{ab})^2}{2\sqrt{3.992ab}} < 1.0011,$$

we may take

$$a_1 = \left(40\Delta + \frac{28}{3}\right)\log\alpha, \quad a_2 = 4\log\alpha + 0.044.$$

Then,

$$\frac{b_1}{a_2} + \frac{b_2}{a_1} < \frac{l/2 + 10\Delta + 7/3}{(40\Delta + 28/3)\log\alpha}$$

which enables us to take

$$h = 4\log\frac{m + 10\Delta + 7/3}{(40\Delta + 28/3)\log\alpha} + 11.913.$$

If $h \leq 28.684$, then

$$m < 66.205(40\Delta + 28/3)\log\alpha.$$

If $h > 28.684$, then Proposition 4.2 with (14) implies that

$$m < \begin{cases} 66.921(40\Delta + 28/3)\log\alpha & \text{for } \Delta \leq 10000; \\ 66.216(40\Delta + 28/3)\log\alpha & \text{for } \Delta \geq 10001. \end{cases} \tag{22}$$

Thus, (22) holds in any case. In the case where $\Delta \leq 10000$, we know by [7, Lemma 2.4] that

$$m > b^{-1/2}c^{1/2} > b^{-1/2}(3.992ab^2)^{1/2} > 1.9979a^{1/2}b^{1/2},$$

which together with (22) shows that

$$\frac{\sqrt{ab}}{\log(s+\sqrt{ac})} < \frac{66.921}{1.9979}(40\Delta + 28/3) < 1.34 \cdot 10^7.$$

Since $s+\sqrt{ac} < 4ab$, one obtains $ab < 3.2 \cdot 10^{17}$. However, $b > 10^{13}$ implies $a < 3.2 \cdot 10^4$, which contradicts $10^{13} < b \le 4a^2 + a + 2\sqrt{a}$.

In the case where $\Delta \ge 10001$, since $\beta^2 > \alpha$, comparing the inequality in Lemma 4.5 (1) with (22), one obtains $b/a < 7.035 \cdot 10^6$ and $b < 4.951 \cdot 10^{13}$.

(2) In the case of $\tau = +$, rewrite Λ as

$$\Lambda = \log\left(\alpha^{\Delta}\chi\right) - \frac{l}{2}\log\frac{\alpha}{\beta^2},$$

and assume that $b > 10^{13}$. We apply Proposition 4.2 with $b_1 = l/2$, $b_2 = 1$, $\alpha_1 = \alpha/\beta^2$, $\alpha_2 = \alpha^{\Delta}\chi$. It is easy to see that

$$h(\alpha_1) = \frac{1}{4}\log(s+\sqrt{ac})^2 = \frac{1}{2}\log\alpha.$$

Since $4ab^2 < c < 4.0064ab^2$, $b(\sqrt{a}+\sqrt{b})^2 < 1.0012b^2 < c/(3.9952a)$ and $b < 4.001a^2$, one sees from (21) that

$$h(\chi) < \frac{1}{4}\log\left(\frac{c^3}{3.9952a}\right) < \frac{7}{6}\log\alpha,$$

which yields

$$h(\alpha_2) = h(\alpha^{\Delta}\chi) \le \frac{\Delta}{2}\log\alpha + h(\chi) < \frac{3\Delta+7}{6}\log\alpha.$$

Thus, a_1 and a_2 satisfy

$$\begin{aligned} a_1 &\ge (\rho-1)\log\left(\frac{\alpha}{\beta^2}\right) + 4\log\alpha, \\ a_2 &\ge \left(\Delta(\rho+3) + \frac{28}{3}\right)\log\alpha + (\rho-1)\log\chi. \end{aligned}$$

Fix $\rho = 37$ and $\mu = 0.63$. Since

$$\frac{\alpha}{\beta^2} < \frac{s+\sqrt{ac}}{4ab} < 1.0008$$

and

$$(\rho-1)\log\chi \le 36\log\frac{1+\sqrt{a/b}}{1-\sqrt{a/c}} < 36\log(1.001) < 0.001\log\alpha,$$

we may take

$$a_1 = 8\log\beta + 0.032 \quad \text{and} \quad a_2 = (40\Delta + 9.335)\log\alpha.$$

Then, $l\log\beta < m\log\alpha$ shows that

$$\frac{b_1}{a_2} + \frac{b_2}{a_1} < \frac{l}{2(40\Delta + 9.335)\log\alpha} + \frac{1}{8\log\beta} < \frac{4m + 40\Delta + 9.335}{8(40\Delta + 9.335)\log\beta},$$

which allows us to take

$$h = 4\log\frac{4m + 40\Delta + 9.335}{8(40\Delta + 9.335)\log\beta} + 11.913.$$

If $h \le 28.684$, then

$$m < 132.41(40\Delta + 9.335)\log\beta.$$

If $h > 28.684$, then Proposition 4.2 with (14) implies that

$$m < \begin{cases} 132.789(40\Delta + 9.335)\log\beta & \text{for } \Delta \le 10000; \\ 132.413(40\Delta + 9.335)\log\beta & \text{for } \Delta \ge 10001. \end{cases} \tag{23}$$

Thus, (23) holds in any case. In the case where $\Delta \le 10000$, using (23) and $m > b^{-1/2}c^{1/2} > 2a^{1/2}b^{1/2}$ from [7, Lemma 2.4], one has

$$\frac{2\sqrt{ab}}{\log(r + \sqrt{ab})} < 132.789(40\Delta + 9.335) < 5.312 \cdot 10^7,$$

which yields $ab < 3.1 \cdot 10^{17}$. However, $b > 10^{13}$ implies $a < 3.1 \cdot 10^4$, which contradicts $10^{13} < b \le 4a^2 + a + 2\sqrt{a}$.

In the case where $\Delta \ge 10001$, the inequality in Lemma 4.5 (2) together with (23) implies $b/a < 7.018 \cdot 10^6$ and $b < 4.929 \cdot 10^{13}$. This completes the proof of Proposition 4.6. $\square$

Second consider case (B), in which $m \equiv n \equiv l \equiv 1 \pmod 2$ and $(x_2, y_2) = (1, \tau)$ hold. We appeal to the strategy in [22].

Lemma 4.7. *Assume that $m \equiv n \equiv 1 \pmod 2$. Then, $m \equiv n \equiv \pm 1 \pmod r$.*

Proof. Note that $x = V_l = W_m$, where

$$V_0 = x_2 = 1, \quad V_1 = rx_2 + ay_2 = r + \tau a, \quad V_{l+2} = 2rV_{l+1} - V_l,$$
$$W_0 = r, \quad W_1 = rs + \varepsilon_2 at, \quad W_{m+2} = 2sW_{m+1} - W_m.$$

Since l is odd, one has $x = V_l \equiv (-1)^{\frac{l-1}{2}}\tau a \pmod r$. On the other hand, since $s = 2r(r + \tau a) - 1 \equiv -1 \pmod r$ and $t = 2r(b + \tau r) - \tau \equiv -\tau \pmod r$, one has

$$x = W_m \equiv -\tau\epsilon_2 ma \pmod r.$$

Since $\gcd(a, r) = 1$, one obtains $m \equiv (-1)^{\frac{l+1}{2}}\varepsilon_2 \pmod r$. Similarly, one can deduce from $y = u_l' = u_n$ that

$$u_l' \equiv (-1)^{\frac{l-1}{2}} b \pmod r \quad \text{and} \quad u_n \equiv -\varepsilon_2 nb \pmod r,$$

which together give $n \equiv (-1)^{\frac{l+1}{2}}\varepsilon_2 \pmod r$. It follows that $m \equiv n \equiv \pm 1 \pmod r$. □

Proposition 4.8. *Let $\{a, b, c, d\}$ be a Diophantine quadruple with $a < b < c = c_2^\tau < d_+ < d$ and $\tau \in \{\pm\}$. Assume that $z = v_m = w_n$ for some odd m and n with $|z_0| = t$, $|z_1| = s$ and $z_0 z_1 > 0$. Then, $b < 6.76 \cdot 10^6$.*

Proof. For this proof we may assume $b \geq 10^6$, which entails $a \geq 500$ and $r > 22360$. By Lemma 4.7, one may put

$$m = m_0 r + \varepsilon_0 \quad \text{and} \quad n = n_0 r + \varepsilon_0$$

for some positive integers m_0 and n_0, and $\varepsilon_0 \in \{\pm 1\}$. Putting further

$$\alpha = s + \sqrt{ac}, \quad \beta' = t + \sqrt{bc}, \quad \chi' = \frac{\sqrt{b}(r\sqrt{c} + \varepsilon_2 t\sqrt{a})}{\sqrt{a}(r\sqrt{c} + \varepsilon_2 s\sqrt{b})},$$

one has

$$\begin{aligned} \Lambda' :&= m\log\alpha - n\log\beta' + \log\chi' \\ &= (m_0 r + \varepsilon_0)\log\alpha - (n_0 r + \varepsilon_0)\log\beta' + \log\chi' \\ &= \log\left(\left(\frac{\alpha}{\beta'}\right)^{\varepsilon_0}\chi'\right) - r\log\left(\frac{(\beta')^{n_0}}{\alpha^{m_0}}\right). \end{aligned}$$

We may apply Proposition 4.2 with $b_1 = r$, $b_2 = 1$, $\alpha_1 = (\beta')^{n_0}/\alpha^{m_0}$, $\alpha_2 = (\alpha/\beta')^{\varepsilon_0}\chi'$, by replacing α_1, α_2 with $1/\alpha_1$, $1/\alpha_2$ if $\alpha_1 < 1$ and $\alpha_2 < 1$. We know by [13, formula (60)] that

$$0 < \Lambda' < \frac{8}{3} ac\alpha^{-2m}. \tag{24}$$

Now one has

$$h(\alpha) = \frac{1}{2}\log\alpha, \quad h(\beta') = \frac{1}{2}\log\beta' < \frac{3}{4}\log\alpha$$

and

$$\begin{aligned} h(\chi') &\le \frac{1}{4}\log\left(a^2(c-b)^2\cdot\frac{\sqrt{b}(r\sqrt{c}+t\sqrt{a})}{\sqrt{a}(r\sqrt{c}+s\sqrt{b})}\cdot\frac{\sqrt{b}(r\sqrt{c}+t\sqrt{a})}{\sqrt{a}(r\sqrt{c}-s\sqrt{b})}\cdot\frac{\sqrt{b}(r\sqrt{c}-t\sqrt{a})}{\sqrt{a}(r\sqrt{c}-s\sqrt{b})}\right) \\ &< \frac{1}{4}\log\left(4a^{1/2}b^{3/2}c^2r^2\right) < \frac{11}{8}\log\alpha. \end{aligned}$$

Hence,

$$h(\alpha_2) \le h(\alpha) + h(\beta') + h(\chi') < \frac{21}{8}\log\alpha. \tag{25}$$

If $(\beta')^{n_0} > \alpha^{m_0}$, then the conjugates of $(\beta')^{n_0}/\alpha^{m_0}$ greater than 1 are $(\beta')^{n_0}/\alpha^{m_0}$ and $(\beta')^{n_0}/\overline{\alpha}^{m_0}$, where $\overline{\alpha} = s - \sqrt{ac}$. Thus,

$$h(\alpha_1) = h\left(\frac{(\beta')^{n_0}}{\alpha^{m_0}}\right) = \frac{1}{4}\log\left((\beta')^{2n_0}\right) = \frac{n_0}{2}\log\beta'.$$

If $(\beta')^{n_0} < \alpha^{m_0}$, then similarly one has $h(\alpha_1) = (m_0/2)\log\alpha$. By (24), one gets

$$\begin{aligned} \left|\frac{m_0}{2}\log\alpha - \frac{n_0}{2}\log\beta'\right| = \frac{1}{2}\left|\log\frac{\alpha^{m_0}}{(\beta')^{n_0}}\right| &< \frac{1}{2r}\left(|\log\left((\alpha/\beta')^{\varepsilon_2}\chi'\right)| + \frac{8}{3}ac\,\alpha^{-2m}\right) \\ &< \frac{1}{2r}\left(\log\alpha + 10^{-10}\right) < 0.00048, \end{aligned}$$

where we used $3.74865ab^2 < c_2^- \le c \le c_2^+ < 4.25952ab^2$, $m \ge 22360$, and

$$\begin{aligned} \log\chi' &< \log(1.5b^{1/2}a^{-1/2}) < 0.5\log\alpha, \\ |\log\left((\alpha/\beta')^{\varepsilon_2}\chi'\right)| &\le \log(\beta'\chi'/\alpha) < \log\alpha, \\ \frac{1}{2r}\log\alpha &< \frac{\log\left(\sqrt{4.25952a^2b^2+1}+\sqrt{4.25952}ab\right)}{2\sqrt{ab+1}} < 0.0004796. \end{aligned}$$

Thus, in either case one has

$$h(\alpha_1) < \frac{m_0}{2}\log\alpha + 0.00048. \tag{26}$$

According to Proposition 4.2, we may take

$$\begin{aligned} a_1 &= 4m_0\log\alpha + 0.00096(\rho+3), \\ a_2 &= (\rho+20)\log\alpha, \\ b' &= \frac{r}{a_2} + \frac{1}{a_1}, \\ h &= 4\log b' + 4\log\lambda + 7.06. \end{aligned}$$

Fix $\rho = 37$ and $\mu = 0.63$. If $h \le 28.77$ then $b' < 67.657$, so that

$$r < 57 \cdot 67.657 \log \alpha < 3856.449 \log \alpha.$$

If $h > 28.77$ then Proposition 4.2 together with (24) implies that

$$2m \log \alpha < 33.85945 \cdot 57(4m_0 \log \alpha + 0.0384) \log \alpha + \log(8ab/3).$$

As $m \geq m_0 r - 1$, one gets

$$r < 3864.828 \log \alpha.$$

Thus, one always has $ab < 8.7839 \cdot 10^9$, $r \leq 93722$, and $b < 6.76 \cdot 10^6$. $\square$

Now, starting with the bounds obtained in Propositions 4.6 and 4.8, one can apply the reduction method to get a contradiction in each case. The computations needed in all about 23 days.

4.3. The case $c \geq c_3^-$

The following lemmas can be applied if c is much bigger than b.

Lemma 4.9. ([20, Lemma 4.1]) *Assume that $c > b^4$. Suppose that $z = v_m = w_n$ has a solution for some integers m and n. Then, $m \equiv n \pmod 2$ and $n > 2.778b^{-3/4}c^{1/4}$.*

Lemma 4.10. ([20, Lemma 3.4]) *Assume that $c \geq 3.706b^4$. If $z = v_m = w_n$ has a solution for some integers m and n with $n \geq 4$, then $n < 8\varphi(a, b, c)$, where*

$$\varphi(a, b, c) = \frac{\log(8.406 \cdot 10^{13} a^{1/2}(a')^{1/2} b^2 c) \log(1.643 a^{1/2} b^{1/2} (b-a)^{-1} c)}{\log(4bc) \log(0.2699 a (a')^{-1} b^{-1} (b-a)^{-2} c)}$$

and $a' = \max\{a, b - a\}$.

Suppose first that (3) holds. Assume that $\{a, b, c, d\}$ is a Diophantine quadruple with $c_3^- \leq c < d_+ < d$. Then $c > 16a^2b^2(a + b - 2r)$. Since inequalities (3) and $b > 4000$ together imply that $a \geq 37$, $0.923a^2 < b$ and

$$\begin{aligned} a + b - 2r = \frac{(b-a)^2 - 4}{a + b + 2r} &> \frac{\left\{\left(1 - (0.923b)^{-1/2}\right)^2 - 4b^{-2}\right\} b}{1 + (0.923b)^{-1/2} + 2\sqrt{(0.923b)^{-1/2} + b^{-2}}} \\ &> 0.759b, \end{aligned}$$

from $b \leq 2.98a^2$ it follows that $c > 16 \cdot (1/2.98) \cdot 0.759b^4 > 4.0751b^4$. Since $(b/2.98)^{1/2} \leq a < (b/0.923)^{1/2}$ and $a^{1/2}b^{1/2}(b - a)^{-1} < 1.037306b^{-1/4}$, we see from Lemmas 4.9 and 4.10 that

$$2.778b^{-3/4}c^{1/4} < \frac{8\log(8.5761\cdot 10^{13}b^{11/4}c)\log(1.7043b^{-1/4}c)}{\log(4bc)\log(0.15634b^{-7/2}c)},$$

which together with $c > 4.0751b^4$ implies that

$$3.9469b^{1/4} < \frac{8\log(3.4949\cdot 10^{14}b^{27/4})\log(6.9452b^{15/4})}{\log(16.3004b^5)\log(0.6371b^{1/2})}.$$

Hence, we obtain $b \le 800822$, $a \le 908$ and $r \le 26965$. The reduction procedure gives $m \le 3$, so that comparison to (15) leads to

$$3 \ge m > 2\sqrt{\frac{b}{a}}\log(2\sqrt{ab}) > \log 16000 > 9.6,$$

a contradiction.

Suppose second that $2.98a^2 < b \le 4a^2 + a + 2\sqrt{a}$, and that $c = c_\nu^\tau$ for some ν and τ. Since $4a^2 + a + 2\sqrt{a} < 4.01a^2$ and $a + b - 2r > 0.93b$, it holds

$$c \ge c_3^- > 16a^2b^2(a+b-2r) > \frac{16}{4.01}b^3 \cdot 0.93b > 3.71b^4. \tag{27}$$

Lemmas 4.9 and 4.10, together with

$$a^{1/2} < \left(\frac{b}{2.98}\right)^{1/4} < 0.76111b^{1/4}, \quad (b-a)^{-1} = \frac{1}{b}\left(1 + \frac{a}{b-a}\right) < 1.00127b^{-1}$$

and $b < 4.01a^2$, imply that

$$2.778b^{-3/4}c^{1/4} < \frac{8\log(6.3979\cdot 10^{13}b^{11/4}c)\log(1.2521b^{-1/4}c)}{\log(4bc)\log(0.13478b^{-7/2}c)}. \tag{28}$$

Combining the estimates (27) and (28) yields $b \le 969729$, $a \le 570$ and $r \le 23510$. The reduction procedure then leads us in less than 16 minutes to a contradiction as indicated above.

5. Proof of Theorem 1.3

Suppose finally that $a = T^2 + 2T$ and $b = 4T^4 + 8T^3 - 4T$ for some integer $T > 1$. We know by $b > 4000$ that $T \ge 6$. Moreover, the proof of Lemma 3.1 implies that it suffices to consider the case where c can be expressed as $c = \gamma_\nu^\tau$ for some sequence $\{\gamma_\nu^\tau\}$ satisfying $\gamma_{\nu_0}^{\tau_0} = c' \in \{1, T^2 - 1\}$ with some ν_0 and τ_0. Let us show the sequence $\{\gamma_\nu^\tau\}$ explicitly. Any positive solution to the Pellian equation

$$at^2 - bs^2 = a - b \tag{29}$$

can be written as $s = \sigma_\nu$, where

$$\sigma_0 = s_0, \quad \sigma_1 = rs_0 + at_0, \quad \sigma_{\nu+2} = 2r\sigma_{\nu+1} - \sigma_\nu,$$

for some solution (t_0, s_0) to (29) with $1 \le s_0 < \sqrt{(r+1)/2} = T\sqrt{T+2}$ by [12, Lemma 1]. Now, $\sigma_2 > r$ implies $(\sigma_2^2 - 1)/a > b > c'$, and $\sigma_0 = s_0 < T\sqrt{T+2}$ implies $(\sigma_0^2 - 1)/a < T$. Hence, one must have either $(\sigma_i^2 - 1)/a = 1$ with $i \in \{0, 1\}$ or $(\sigma_1^2 - 1)/a = T^2 - 1$.

If $(\sigma_0^2 - 1)/a = 1$, then one has $s_0 = \sigma_0 = T + 1$ and $t_0 = \pm(2T^2 + 2T - 1)$.

If $(\sigma_1^2 - 1)/a = 1$, then $rs_0 + at_0 = \sigma_1 = T + 1$ with $t_0 < 0$. Since $at_0^2 - bs_0^2 = a - b$, one has $s_0 = T^2 + T - 1$ and $t_0 = -2T^3 - 2T^2 + 2T + 1$.

If $(\sigma_1^2 - 1)/a = T^2 - 1$, then $rs_0 + at_0 = \sigma_1 = T^2 + T - 1$ with $t_0 < 0$, and hence $s_0 = T+1$, $t_0 = -2T^2 - 2T + 1$. Thus, one sees that the sequence $\{\gamma_\nu^\tau\}$ obtained from the first case $(t_0, s_0) = (\pm(2T^2 + 2T - 1), T + 1)$ contains the other possible values. Therefore, one may write

$$\begin{aligned}
&\gamma_0 = \gamma_0^\tau = 1, \ \gamma_1^- = T^2 - 1, \\
&\gamma_1^+ = 16T^6 + 64T^5 + 72T^4 - 31T^2 - 4T + 3, \\
&\gamma_2^- = 16T^8 + 64T^7 + 48T^6 - 80T^5 - 88T^4 + 32T^3 + 36T^2 - 4T - 3, \\
&\gamma_2^+ = 256T^{12} + 2048T^{11} + 6272T^{10} + 8448T^9 + 2576T^8 - 5248T^7 - 4464T^6 \\
&\qquad + 560T^5 + 1400T^4 + 96T^3 - 156T^2 - 12T + 5, \\
&\gamma_3^- > 256T^{14} + 2048T^{13}
\end{aligned}$$

(where the indices of the sequence are assigned in correspondence with $\{\sigma_\nu\}$).

Since $b < c$, it suffices to consider $c \ge \gamma_1^+$.

Firstly, consider the case where $c = \gamma_1^+$. As usual, let $z = v_m$ and $z = w_n$ be solutions to Pellian equations (9) and (10), respectively. Then, as seen at the beginning of Subsection 4.2, we may assume that the parities of m and n are the same; more precisely, either (A) or (B) holds. Let $y = u_n$ and $y = u'_l$ be solutions to Pellian equations (10) and (8), respectively. Then, the following holds.

Lemma 5.1. *Let $c = \gamma_1^+$. Assume that $z = v_m = w_n$ holds for some positive integers m and n. Then, $m \equiv n \equiv l \equiv 0 \pmod 2$ with $z_0 = z_1 = \varepsilon_1 \in \{\pm 1\}$ and $x_2 = y_2 = 1$.*

Proof. In the case where $m \equiv n \equiv 0 \pmod 2$ with $z_0 = z_1 = \varepsilon_1 \in \{\pm 1\}$, the argument used in the proof of Lemma 4.1 shows that $l \equiv 0 \pmod 2$ and $x_2 = y_2 = 1$.

Assume that $m \equiv n \equiv 1 \pmod 2$ with $z_0 = \varepsilon_2 t$, $z_1 = \varepsilon_2 s$, $\varepsilon_2 \in \{\pm 1\}$. Then, the sequence $\{u_n\}$ is given by (18). Since both n and $s = 4T^4 + 12T^3 + 7T^2 - 3T - 1$ are odd and $t = 8T^5 + 24T^4 + 14T^3 - 10T^2 - 6T + 1$, we have

$$u_n \equiv b + rt \equiv b + 2T^2 + 2T - 1 \pmod{2b}. \tag{30}$$

If l is even, then (12) and (30) together imply $b + 2T^2 + 2T - 1 \equiv y_2 \pmod{2b}$. However, since $b < b + 2T^2 + 2T - 1 < 2b$ and $|y_2| < \sqrt{b\sqrt{b}} < b$ by (11), it would hold

$2T^2+2T-1=b+y_2$ and $y_2=-4T^4-8T^3+2T^2+6T-1$, which do not satisfy (11). Thus, l is odd. Then (12) and (30) together yield

$$b+2T^2+2T-1\equiv \pm(bx_2-r|y_2|) \pmod{2b}.$$

Since $0<bx_2-r|y_2|<b$, it holds

$$2T^2+2T-1=b-bx_2+r|y_2|.$$

Then, one sees from $ay_2^2-bx_2^2=a-b$ that

$$\begin{aligned} &x_2^2+2(T^2+2T)(4T^4+8T^3-2T^2-6T+1)x_2\\ &\quad-(8T^6+32T^5+27T^4-24T^3-25T^2+6T+1)=0. \end{aligned}$$

The squareness of the discriminant of this quadratic equation leads to

$$4T^6+16T^5+12T^4-16T^3-13T^2+6T+1=X^2$$

with a positive integer X. However, we then have

$$(2T^3+4T^2-T-2)^2<X^2<(2T^3+4T^2-T-1)^2,$$

which is absurd. This completes the proof of Lemma 5.1. $\square$

In view of Lemma 5.1, we may consider only case (A), and hence have the lower bound $m>b^{-1/2}c^{1/2}$. Certainly, this bound together with Baker's method gives an upper bound for T. However, the bound obtained in this way is too large to apply the reduction method for each T. Thus, we improve the lower bound for m above, by using the fact that both a and b are divisible by T.

Lemma 5.2. *Let $c=\gamma_1^+$. Assume that $T>10^{10}$. If $v_m=w_n$ holds for some positive integers m and n, then $m>3.9999b^{-1/2}c^{1/2}T^{1/2}$.*

Proof. Since $z_0=z_1=\varepsilon_1\in\{\pm1\}$, the sequences $\{v_m\}$ and $\{w_n\}$ are given by

$$\begin{aligned} &v_0=\varepsilon, \quad v_1=c+\varepsilon s, \quad v_{m+2}=2sv_{m+1}-v_m,\\ &w_0=\varepsilon, \quad w_1=c+\varepsilon t, \quad w_{n+2}=2tw_{n+1}-w_n. \end{aligned}$$

Considering v_m and w_n modulo $8c^2T$, we see from $m\equiv n\equiv 0 \pmod 2$ that

$$\begin{aligned} v_m&\equiv mcs+\varepsilon\frac{m^2}{2}ac+\varepsilon \pmod{8c^2T},\\ w_n&\equiv nct+\varepsilon\frac{n^2}{2}bc+\varepsilon \pmod{8c^2T}, \end{aligned}$$

which together imply that

$$\varepsilon\left(a\cdot\frac{m^2}{2}-b\cdot\frac{n^2}{2}\right)\equiv tn-sm \pmod{8cT}. \tag{31}$$

Suppose now that $m\le 3.9999b^{-1/2}c^{1/2}T^{1/2}$. Since $n\le m\le 2n$ by [13, Lemma 3] with m, n even, one gets

$$\left|a\cdot\frac{m^2}{2}-b\cdot\frac{n^2}{2}\right|<b\cdot\frac{m^2}{2}\le 7.9999600005cT,$$

$$|tn-sm|<tm\le 3.9999cT^{1/2}\sqrt{1+\frac{1}{bc}}<0.000039999cT,$$

where the last inequality holds because $T\ge 10^{10}$. Hence, congruence (31) is in fact an equality. Multiplying both sides of it by $tn+sm$ one has

$$\left(b\cdot\frac{n^2}{2}-a\cdot\frac{m^2}{2}\right)(2c+\varepsilon(tn+sm))=m^2-n^2.$$

If $m^2-n^2=0$, then $m=n$. Since $b>a$, one gets $2c=m(t+s)$. However, $c\equiv 3 \pmod{T}$ and $t+s\equiv 0 \pmod{T}$ together imply $6\equiv 0 \pmod{T}$, which contradicts $T>10^{10}$. Thus, $2c+\varepsilon(tn+sm)\ne 0$ and

$$|m^2-n^2|\ge\left|b\cdot\frac{n^2}{2}-a\cdot\frac{m^2}{2}\right|,$$

which yields,

$$\left|\frac{b}{a}-\frac{m^2}{n^2}\right|\le\frac{2|m^2/n^2-1|}{a}.$$

However, this inequality is not compatible with the inequalities

$$\frac{b}{a}\ge 3.9\cdot 10^{20}\quad(\text{by } T>10^{10})\quad\text{and}\quad\frac{m^2}{n^2}\le 4.$$

Therefore, one obtains $m>3.9999b^{-1/2}c^{1/2}T^{1/2}$. $\square$

Now if we apply Aleksentsev's theorem [1, Theorem 1] to the linear form

$$\Lambda=m\log\alpha-n\log\beta'-\log\chi',$$

where $\alpha=s+\sqrt{ac}$, $\beta'=t+\sqrt{bc}$ and

$$\chi'=\frac{\sqrt{b}(\sqrt{c}+\varepsilon\sqrt{a})}{\sqrt{a}(\sqrt{c}+\varepsilon\sqrt{b})}$$

with $\varepsilon \in \{\pm 1\}$ (χ' is redefined here and hopefully is not mistakenly identified with the quantity introduced in the proof of Proposition 4.8), and combine the estimate $0 < \Lambda' < (8ac/3)\alpha^{2m}$ given by [13, Eq. (60)] with $m > 3.9999b^{-1/2}c^{1/2}T^{1/2}$, then we would get around $T < 1.8 \cdot 10^{11}$. In order to further reduce this upper bound for T, we will appeal to Matveev's theorem ([27, Theorem 2.1]). To do that, we have to check the following.

Lemma 5.3. *Assume that none of the following is a square in $\mathbb{Q}$:*

$$2(s \pm 1),\ 2(t \pm 1),\ (s \pm 1)(t \pm 1),\ (s \pm 1)(t \mp 1),\ 2ab(s \pm 1),\ 2ab(t \pm 1). \tag{32}$$

Then the numbers $\sqrt{\alpha}$, $\sqrt{\beta'}$, $\sqrt{\chi'}$ satisfy the Kummer condition with respect to the field $K := \mathbb{Q}(\sqrt{ac}, \sqrt{bc})$, that is,

$$[K(\sqrt{\alpha}, \sqrt{\beta'}, \sqrt{\chi'}) : K] = 2^3 = 8.$$

Proof. Firstly, we show that $\sqrt{\alpha} \notin K$. Assume on the contrary that $\sqrt{\alpha} \in K$. Then, one may write

$$\sqrt{\alpha} = l_0 + l_1\sqrt{ab} + l_2\sqrt{ac} + l_3\sqrt{bc}$$

with l_0, l_1, l_2, $l_3 \in \mathbb{Q}$. Squaring both sides yields

$$\begin{aligned} s + \sqrt{ac} = {} & l_0^2 + abl_1^2 + acl_2^2 + bcl_3^2 + 2(l_0l_1 + cl_2l_3)\sqrt{ab} \\ & + 2(l_0l_2 + bl_1l_3)\sqrt{ac} + 2(l_0l_3 + al_1l_2)\sqrt{bc}. \end{aligned}$$

Comparing the coefficients of $\sqrt{ab}$ and $\sqrt{bc}$ in both sides, one gets

$$l_0l_1 + cl_2l_3 = l_0l_3 + al_1l_2 = 0.$$

In case $l_2 \neq 0$, it holds $cl_3^2 = al_1^2$. If $l_1l_3 \neq 0$, then ac would be a square, which contradicts $ac + 1 = s^2$. Thus one has $l_1 = l_3 = 0$ and

$$s + \sqrt{ac} = l_0^2 + acl_2^2 + 2l_0l_2\sqrt{ac},$$

which yields $2l_0l_2 = 1$ and $4l_0^4 - 4sl_0^2 + ac = 0$. It follows that $s = 2l_0^2 \pm 1$, which contradicts the assumption.

In case $l_2 = 0$, one gets $2bl_1l_3 = 1$ and $l_0 = 0$, which together show that $4ab^3l_1^4 - 4b^2sl_1^2 + bc = 0$. This means that $s \pm 1 = 2abl_1^2$, which contradicts the assumption. Therefore, one obtains $\sqrt{\alpha} \notin K$.

In the same way as above, one also sees from the assumptions $2(t \pm 1) \neq \square$ and $2ab(t \pm 1) \neq \square$ that $\sqrt{\beta'} \notin K$.

Secondly, assume that $\sqrt{\alpha} \in K(\sqrt{\beta'})$. One may write $\sqrt{\alpha} = k_0 + k_1\sqrt{\beta'}$ for some k_0, $k_1 \in K$. Then,

$$s + \sqrt{ac} = k_0^2 + k_1^2(t + \sqrt{bc}) + 2k_0k_1\sqrt{\beta'}.$$

If $k_0k_1 \neq 0$, then this equation means that $\sqrt{\beta'} \in K$, which is impossible as seen above. If $k_1 = 0$, then $s + \sqrt{ac} = k_0^2$, which contradicts $\sqrt{\alpha} \notin K$. If $k_0 = 0$, then $s + \sqrt{ac} = k_1^2(t + \sqrt{bc})$ with $k_1 = l_0 + l_1\sqrt{ab} + l_2\sqrt{ac} + l_3\sqrt{bc}$, where $l_0, l_1, l_2, l_3 \in \mathbb{Q}$. Explicitly, this means

$$st = l_0^2 + abl_1^2 + acl_2^2 + bcl_3^2, \tag{33}$$

$$c = -2(l_0l_1 + cl_2l_3), \tag{34}$$

$$s = -2(l_0l_3 + al_1l_2), \tag{35}$$

$$t = 2(l_0l_2 + bl_1l_3). \tag{36}$$

If $l_0 = 0$ then $2l_2l_3 = -1$. Elimination of s and t leads to the equality $abl_1^2 = acl_2^2 + bcl_3^2$. The arithmetic mean-geometric mean inequality shows that the right-hand side is at least $2|l_2l_3|c\sqrt{ab} = c\sqrt{ab}$. This implies $st = 2abl_1^2 \geq 2c\sqrt{ab}$, which readily leads to the contradiction $bc \geq abc^2$. The same contradiction is obtained assuming $l_1l_2l_3 = 0$.

Suppose now that it holds $l_0l_1l_2l_3 \neq 0$. Then

$$c = -\frac{2l_0l_1}{1 + 2l_2l_3}.$$

Using this, the elimination of s and t results in the equation

$$(4l_2l_3 + 1)(l_0^2 + abl_1^2) + (c + 4l_0l_1)(al_2^2 + bl_3^2) = 0,$$

equivalently

$$(4l_2l_3 + 1)\big(l_0^2 + abl_1^2 - acl_2^2 - bcl_3^2\big) = 0.$$

In case $4l_2l_3 + 1 = 0$ one gets

$$c = -4l_0l_1, \quad l_2 = \frac{bsl_1 + tl_0}{2(l_0^2 - abl_1^2)}, \quad l_3 = -\frac{atl_1 + bsl_0}{2(l_0^2 - abl_1^2)}.$$

Squaring (35), one obtains $2(l_0l_3 - al_1l_2) = \pm 1$. Combining the last three equations, one finds

$$(s \pm 1)l_0^2 + 2atl_0l_1 + ab(s \mp 1)l_1^2 = 0,$$

which yields

$$l_0 = -\frac{(s \pm 1)(t + 1)}{c}l_1 \quad \text{or} \quad -\frac{(s \pm 1)(t - 1)}{c}l_1.$$

It follows from $c = -4l_0l_1$ that

$$c^2 = 4(s \pm 1)(t + 1)l_1^2 \quad \text{or} \quad 4(s \pm 1)(t - 1)l_1^2,$$

which contradicts the assumption.

It remains to consider the possibility $l_0^2 + abl_1^2 = acl_2^2 + bcl_3^2$, when $st = 2(l_0^2 + abl_1^2) = 2(acl_2^2 + bcl_3^2)$. Since k_1 is an algebraic integer in a quartic field of type $(2, 2)$ over $\mathbb{Q}$, [17, Lemma 3.1] allows us to assume that $4l_i \in \mathbb{Z}$ for $i \in \{0, 1, 2, 3\}$. From $st \ge 4|l_0 l_1|\sqrt{ab} = 2|2l_2l_3 + 1|c\sqrt{ab}$, $st \ge 4|l_2l_3|c\sqrt{ab}$, and $4l_2l_3 \ne -1$, one has

$$st \ge \max\{2|2l_2l_3 + 1|, 4|l_2l_3|\} \cdot c\sqrt{ab} \ge \frac{5}{4}c\sqrt{ab}, \tag{37}$$

where the last equality is attained by $l_2l_3 \in \{-5/16, -3/16\}$. On the other hand, since $a \ge 1$, $b \ge 3$ and $c \ge 8$, one gets

$$st = \sqrt{1 + \frac{1}{ac}}\sqrt{1 + \frac{1}{bc}} \cdot c\sqrt{ab} < 1.1c\sqrt{ab},$$

which contradicts (37).

Hence, $\sqrt{\alpha} \notin K(\sqrt{\beta'})$.

Similarly, one also sees that $\sqrt{\beta'} \notin K(\sqrt{\alpha})$.

Thirdly, assume that $\sqrt{\chi'} \in K$. Since

$$\chi' = \frac{-ab + c\sqrt{ab} - \varepsilon b\sqrt{ac} + \varepsilon a\sqrt{bc}}{a(c - b)}, \tag{38}$$

putting

$$\sqrt{\chi'} = l_0 + l_1\sqrt{ab} + l_2\sqrt{ac} + l_3\sqrt{bc}$$

with some l_0, l_1, l_2, $l_3 \in \mathbb{Q}$, one has

$$\begin{aligned} -ab + c\sqrt{ab} - \varepsilon b\sqrt{ac} + \varepsilon\sqrt{bc} &\\ = a(c - b)\big\{l_0^2 + abl_1^2 + acl_2^2 + bcl_3^2 + 2(l_0l_1 + cl_2l_3)\sqrt{ab}&\\ + 2(l_0l_2 + bl_1l_3)\sqrt{ac} + 2(l_0l_3 + al_1l_2)\sqrt{bc}\big\}&. \end{aligned}$$

Comparing the terms in $\mathbb{Q}$ yields

$$-b = (c - b)(l_0^2 + abl_1^2 + acl_2^2 + bcl_3^2).$$

Since a, b, c, $c - b$ are positive, this is impossible. Hence, $\sqrt{\chi'} \notin K$.

It remains only to show that $\sqrt{\chi'} \notin K(\sqrt{\alpha}, \sqrt{\beta'})$. Assume on the contrary that $\sqrt{\chi'} \in K(\sqrt{\alpha}, \sqrt{\beta'})$ and put

$$\sqrt{\chi'} = k_0 + k_1\sqrt{\alpha} + k_2\sqrt{\beta'} + k_3\sqrt{\alpha\beta'}$$

with some k_0, k_1, k_2, $k_3 \in K$. Squaring both sides, one has

$$\chi' = k_0^2 + k_1^2\alpha + k_2^2\beta' + k_3^2\alpha\beta' + 2(k_0k_1 + k_2k_3\beta')\sqrt{\alpha} + 2(k_0k_2 + k_1k_3\alpha)\sqrt{\beta'} + 2(k_0k_3 + k_1k_2)\sqrt{\alpha\beta'}. \tag{39}$$

In case either $k_0k_1 + k_2k_3\beta' \neq 0$ or $k_0k_3 + k_1k_2 \neq 0$, the equation above means that $\sqrt{\alpha} \in K(\sqrt{\beta'})$, which is a contradiction. In case $k_0k_1 + k_2k_3\beta' = k_0k_3 + k_1k_2 = 0$, if $k_2 \neq 0$, then $k_3^2\beta' = k_1^2$. Since $\sqrt{\beta'} \notin K$, one has $k_1 = k_3 = 0$ and $k_0 = 0$. Similarly one sees that if $k_1 \neq 0$, then $k_0 = k_2 = k_3 = 0$, and if $k_3 \neq 0$, then $k_0 = k_1 = k_2 = 0$. Hence, it suffices to show that none of χ'/α, χ'/β' and $\chi'/(\alpha\beta')$ is a square in K.

Suppose first $\sqrt{\chi'/\alpha} \in K$. One may write

$$\chi' = (s + \sqrt{ac})(l_0 + l_1 + l_2\sqrt{ab} + l_2\sqrt{ac} + l_3\sqrt{bc})^2$$

with some l_0, l_1, l_2, $l_3 \in \mathbb{Q}$. Multiplying both sides by $s - \sqrt{ac}$ and comparing the terms in $\mathbb{Q}$ yield

$$-ab(s - \varepsilon c) = a(c - b)(l_0^2 + abl_1^2 + acl_2^2 + bcl_3^2).$$

Since the right-hand side is positive, one gets $\varepsilon = 1$. On the other hand, since it also holds $\sqrt{\chi'\alpha} \in K$, one may write

$$\chi'(s + \sqrt{ac}) = (l_0' + l_1'\sqrt{ab} + l_2'\sqrt{ac} + l_3'\sqrt{bc})^2$$

with some l_0', l_1', l_2', $l_3' \in \mathbb{Q}$. Comparing the terms in $\mathbb{Q}$ yields

$$-ab(s + \varepsilon c) = a(c - b)((l_0')^2 + ab(l_1')^2 + ac(l_2')^2 + bc(l_3')^2),$$

which shows that $\varepsilon = -1$, in contradiction with our previous finding $\varepsilon = 1$. Hence, $\sqrt{\chi'/\alpha} \notin K$.

In the same way, one sees that $\sqrt{\chi'/\beta'} \notin K$.

Suppose finally that $\sqrt{\chi'/(\alpha\beta')} \in K$, which is equivalent to $\sqrt{\chi'\alpha/\beta'} \in K$. Thus one may write

$$\chi'(s + \sqrt{ac})(t - \sqrt{bc}) = (l_0 + l_1\sqrt{ab} + l_2\sqrt{ac} + l_3\sqrt{bc})^2$$

with some l_0, l_1, l_2, $l_3 \in \mathbb{Q}$. Multiplying both sides by $a(c - b)$, one finds that the term in $\mathbb{Q}$ of the left-hand side is

$$-abst - abc^2 - \varepsilon abct - \varepsilon abcs = -ab(s + \varepsilon c)(t + \varepsilon c) < 0,$$

while the term in $\mathbb{Q}$ of the right-hand side is positive, which is a contradiction. Hence, $\chi'/(\alpha\beta') \notin K$. This completes the proof of Lemma 5.3. $\square$

Recall Matveev's theorem in [27] simplified a little bit to better suit our situation. Let $\alpha_1, \alpha_2, \alpha_3$ be real algebraic numbers and $K := \mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)$. Put $D := [K : \mathbb{Q}]$. Assume that $\alpha_1, \alpha_2, \alpha_3$ satisfy the Kummer condition, that is,

$$[K(\sqrt{\alpha_1}, \sqrt{\alpha_2}, \sqrt{\alpha_3}) : K] = 8.$$

Consider a linear form

$$\Gamma := b_1 \log \alpha_1 + b_2 \log \alpha_2 + b_3 \log \alpha_3,$$

where b_1, b_2, b_3 are integers with $b_3 \neq 0$. Put $A_j := h(\alpha_j)$ for $1 \leq j \leq 3$. We take E, E_1, C_3, C_1, C_2 as follows:

$$\begin{aligned}
E &\geq \frac{1}{3D} \max \left\{ \left| \pm \frac{\log \alpha_1}{A_1} \pm \frac{\log \alpha_2}{A_2} \pm \frac{\log \alpha_3}{A_3} \right| \right\}, \\
E_1 &= \frac{1}{2D} \left(\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} \right), \\
C_3^* \exp(C_3^*) \frac{Ee}{2} &\geq e^3, \quad C_3 = \max\{C_3^*, 3\}, \\
C_1 &= \left(1 + \frac{e^{-6}}{148} \right) (3 \log 2 + 2) \frac{4}{3C_3}, \\
C_2 &= 16 \left(6 + \frac{5}{3 \log 2 + 2} \right) \frac{e^6}{3^{1/2} C_3}.
\end{aligned}$$

We also put

$$\begin{aligned}
\Omega &:= A_1 A_2 A_3, \\
\omega &:= \Omega \left(\frac{DC_1}{e} \right)^3 C_3 \exp(C_3) \frac{Ee}{2}.
\end{aligned}$$

Let C_0 be a real number satisfying

$$C_0 \geq \max \left\{ 2C_3, \log \left(4C_2 \max \left\{ \frac{C_0 \omega}{4C_1 A_3}, C_0, \frac{2E_1 C_3}{C_1} \right\} \right) \right\}.$$

Furthermore, put

$$\begin{aligned}
B_0 &:= \sum_{j=1}^{2} \frac{|b_3| + |b_j|}{8 \gcd(b_j, b_3) C_0 C_2 \omega}, \\
B_1 &:= \sum_{j=1}^{2} \frac{1}{24 \gcd(b_j, b_3) C_1} \left(\frac{|b_3|}{A_j} + \frac{|b_j|}{A_3} \right),
\end{aligned}$$

$$B_2 = \sum_{j=1}^{2} \frac{|\log \alpha_j|(|b_3| + |b_j|)}{8|b_3|C_0C_2\omega},$$

$$B_3 = \sum_{j=1}^{2} \frac{|\log \alpha_j|}{24|b_3|C_1} \left(\frac{|b_3|}{A_j} + \frac{|b_j|}{A_3}\right),$$

and take a real number W_0 satisfying

$$W_0 \geq \max\{2C_3, \log(e(1 + B_0 + B_1 + B_2 + B_3))\}.$$

Now we are ready to state [27, Theorem 2.1] in a form applicable to our situation.

Theorem 5.4. *Suppose that*

$$\begin{aligned} 2\omega \min\{C_0, W_0\} &\geq C_3, \\ \omega \min\{C_0, W_0\} &\geq 2C_1C_3A_j \quad \textit{for } 1 \leq j \leq 3, \\ 3(4C_1)^2 4C_0\Omega &\geq C_3A_j \quad \textit{for } 1 \leq j \leq 3. \end{aligned}$$

Then,

$$\log|\Gamma| > -11648C_2C_0W_0\omega.$$

In order to apply this result to the case where

$$a = T^2 + 2T,\ b = 4T^4 + 8T^3 - 4T,\ c = \gamma_1^+ = 16T^6 + 64T^5 + 72T^4 - 31T^2 - 4T + 3,$$

we need to check that the hypothesis of Lemma 5.3 holds. Since

$$\begin{aligned} a = T(T+2),\ b &= 4T(T+1)(T^2+T-1), \\ s-1 &= (T+1)(T+2)(2T-1)(2T+1), \\ s+1 &= T(2T+3)(2T^2+3T-1), \\ t-1 &= 2T(2T+1)(2T+3)(T^2+T-1), \\ t+1 &= 2(T+1)^2(2T-1)(2T^2+3T-1), \end{aligned}$$

it is not difficult to see that among (32) the only square values for small T are $2(s-1)$ for $T = 1$, $2(t+1)$ for $T \in \{1, 5\}$ and $(s-1)(t+1)$ for $T = 1$. The next lemma shows that no other square appears for bigger T.

Lemma 5.5. *For any $T \geq 6$, none of the following is a square in $\mathbb{Q}$:*

$$2(s \pm 1),\ 2(t \pm 1),\ (s \pm 1)(t \pm 1),\ (s \pm 1)(t \mp 1),\ 2ab(s \pm 1),\ 2ab(t \pm 1).$$

Proof. The idea is to exploit information on the gcd of factors in order to reduce the question to determination of integral points on certain elliptic curves, for which task either Magma [5] or Sage [30] can be used. We used both packages and noticed no difference in their outputs. Occasionally, elementary arguments provide a shorter way to the desired conclusion.

Case 1. Assume $2(s+1)$ is a perfect square, which is denoted shortly $2(s+1) = \square$. Explicitly, $2T(2T+3)(2T^2+3T-1) = \square$. Note that $\gcd\big(2T+3, 2(2T^2+3T-1)\big) = 1$ and $\gcd(T, 2T+3)$ divides 3. If T and $2T+3$ are coprime, then $2T(2T^2+3T-1) = y^2$ for some integer y. With transformation $Y = 4y$, $X = 4T$ one sees the solutions to this equation give rise to integral points on the elliptic curve $Y^2 = X^3 + 6X^2 - 8X$. The computer finds the following integral points on this curve: $(-4, \pm 8)$, $(0, 0)$, $(2, \pm 4)$, $(1250, \pm 44300)$. None of them has the x-coordinate positive and multiple of 4.

When $\gcd(T, 2T+3) = 3$, one has $T = 3u$, $2u+1 = \square$, and $2u(18u^2+9u-1) = \square$. Proceedings as above, one arrives at the conclusion that the integrals point on the elliptic curve $(36y)^2 = (36u)^3 + 18(36u)^2 - 72(36u)$ are $(-18, \pm 36)$, $(0, 0)$, $(4, \pm 8)$. Clearly, none of them has the x-coordinate positive and divisible by 36.

Case 2. Now $2(s-1) = \square$, or $2(T+1)(T+2)(2T+1)(2T-1) = \square$. Having in view that $\gcd\big(T+1, (T+2)(2T+1)\big) = 1$ and $\gcd(T+1, 2T-1) = \gcd(T+1, 3)$, one discusses separately four subcases.

If $T \equiv 1, 3 \pmod 6$ then $T+1 = 2\square$ and $y^2 = (T+2)(2T+1)(2T-1)$, that is $(4y)^2 = (4T)^3 + 8(4T)^2 - 4(4T) - 32$. The only integral points on this elliptic curve are $(-8, 0)$, $(-6, \pm 8)$, $(-3, \pm 5)$, $(-2, 0)$, $(2, 0)$, $(4, \pm 12)$, $(22, \pm 120)$, $(10084, \pm 1013028)$, so that $T \in \{1, 2521\}$. None of these values is acceptable, because $2522 \neq 2\square$.

When $T \equiv 0, 4 \pmod 6$ one has $T+1$ square and $y^2 = 2(T+2)(2T+1)(2T-1) = (2T)^3 + 4(2T)^2 - (2T) - 4$. Sage /Magma readily gives that the only integral points on this elliptic curve are $(-4, 0)$, $(-1, 0)$, and $(1, 0)$. None of them has positive y-coordinate.

For $T \equiv 2 \pmod 6$ one has $T+1 = 3\square$ and $3y^2 = 2(T+2)(2T+1)(2T-1)$, equivalently $(9y)^2 = (6T)^3 + 12(6T)^2 - 9(6T) - 108$. The complete list of integral points is $(-12, 0)$, $(-3, 0)$, $(3, 0)$. As before, no integral point has positive y-coordinate.

The last possibility is $T \equiv 5 \pmod 6$. Then $T+1 = 3\square$ and $3y^2 = (T+2)(2T+1)(2T-1)$, which is transformed into $(36y)^2 = (12T)^3 + 24(12T)^2 - 36(12T) - 864$. Applying the inverse transformation to the outcome list $(-24, 0)$, $(-14, \pm 40)$, $(-12, \pm 36)$, $(\pm 6, 0)$, $(21, \pm 135)$, $(30, \pm 216)$, $(1446, \pm 55440)$, one finds no integer value for T.

Case 3. If $2(t+1)$ is a square, that is, $(2T-1)(2T^2+3T-1) = \square$, then both $2T-1$ and $2T^2+3T-1$ are squares, because $\gcd(2T-1, 2T^2+3T-1) = 1$. Putting $2T-1 = X^2$, one has $2Y^2 = X^4 + 5X^2 + 2$, which is transformed, via $x = 2X^2$ and $y = 4XY$, into the elliptic curve given by

$$C : y^2 = x^3 + 10x^2 + 8x.$$

The integral points on C are $(-9,\pm 3)$, $(-8,\pm 8)$, $(-4,\pm 8)$, $(-2,\pm 4)$, $(-1,\pm 1)$, $(0,0)$, $(2,\pm 8)$, $(4,\pm 16)$, $(18,\pm 96)$, $(32,\pm 208)$, and $(128,\pm 1504)$. Thus, one sees that $X \in \{1,3,4,8\}$, which yields $T \in \{1,5\}$.

Case 4. If $2(t-1)=\square$, then $P := T(2T+1)(2T+3)(T^2+T-1)=\square$. As $\gcd\bigl(2T+1, T(2T+3)\bigr)=1$ and $\gcd\bigl(2T+1, T^2+T-1\bigr) \in \{1,5\}$, there are two subcases to discuss. Suppose first that $2T+1=5\square$. Then $T \equiv 2 \pmod{20}$, which implies $T(2T+3)(T^2+T-1) \equiv 10 \pmod{20}$. This congruence is incompatible with $P/(2T+1)=5\square$.

Assume now that $\gcd\bigl(2T+1, T^2+T-1\bigr)=1$. Then $T(2T+3)(T^2+T-1)=\square$. When $T \not\equiv 0 \pmod 3$, it follows that both T and $(2T+3)(T^2+T-1)$ are squares. The last condition readily implies $(2y)^2=(2T)^3+5(2T)^2+2(2T)-12$. As this elliptic curve has only one integral point, viz., $(-3,0)$, one concludes that it holds $T=3u^2$ and $(2u^2+1)(9u^4+3u^2-1)=3y^2$. Either of the packages Magma, Sage finds only one integral point on the attached elliptic curve $(48y)^2=(24u^2)^3+20(24u^2)^2+32(24u^2)-768$, namely $(-12,0)$. It is not acceptable because its x-coordinate is negative.

Case 5. Suppose $(s+1)(t+1)=\square$, which is equivalent to $y^2=2T(2T+3)(2T-1)=(2T)^3+2(2T)^2-3(2T)$. It is found that none of the integral points on this curve $(-3,0)$, $(-1,\pm 2)$, $(0,0)$, $(1,0)$, $(3,\pm 6)$ has even, positive x-coordinate.

Case 6. When $(s+1)(t-1)=\square$, then $2(2T+1)(2T^2+3T-1)(T^2+T-1)=\square$. One has $\gcd\bigl(2T+1, 2(2T^2+3T-1)\bigr)=1$, $\gcd\bigl(T^2+T-1, 2(2T^2+3T-1)\bigr)=1$, and $\gcd(2T+1, T^2+T-1)=\gcd(2T+1,5)$. When $2T+1$ is coprime with any other factor, it must be a perfect square, as well as T^2+T-1. The last condition implies $(2T+1)^2-5=\square$, so that $T=1$.

If $\gcd(2T+1, T^2+T-1)=5$, then $2T^2+3T-1=2y^2$. Writing the last condition as $(4T+3)^2-17=(4y)^2$, one sees that it implies $4T+3=9$, which is not true for integer T.

Case 7. Assume $(s-1)(t+1)=\square$, explicitly

$$P := 2(T+1)(T+2)(2T+1)(2T^2+3T-1)=\square.$$

One finds $\gcd\bigl(2T+1, 2(T+1)(2T^2+3T-1)\bigr)=1$ and $\gcd(2T+1, T+2)=\gcd(T+2,3)$. Hence, for $2T+1$ coprime with any other factor, one has $2T+1=\square$ and T even, which in turn implies $\gcd\bigl(T+1, 2(T+2)(2T^2+3T-1)\bigr)=1$. Therefore, $y^2=2(T+2)(2T^2+3T-1)$, whence $(4y)^2=(4T)^3+14(4T)^2+40(4T)-64$. The relevant elliptic curve contains the sole integral point $(-8,0)$, whose x-coordinate is negative.

When $\gcd(2T+1, T+2)=3$, then one necessarily has $y^2=2(T+1)(2T^2+3T-1)$. This time one finds on the associated elliptic curve $(4y)^2=(4T)^3+10(4T)^2+16(4T)-32$ the points $(-6,\pm 4)$, $(-4,0)$, $(4,\pm 16)$, the last of which corresponds to $T=1$.

Case 8. Assume that $(s-1)(t-1)$ is a square. Since

$$\gcd(2T(T+1)(T+2)(2T-1)(2T+3), T^2+T-1)=1,$$

one has $T^2 + T - 1 = \square$. Since $4(T^2 + T - 1) = (2T + 1)^2 - 5$, this implies that $T = 1$, a value outside the range of interest.

Case 9. If $2ab(s + 1)$ is a square, then it holds

$$2T(T + 1)(T + 2)(2T + 3)(T^2 + T - 1)(2T^2 + 3T - 1) = \square.$$

On noting that $T^2 + T - 1$ is coprime with any other factor in this product, it results that $T^2 + T - 1$ is a perfect square. As seen in Case 8, this is not possible.

Case 10. If $2ab(s - 1) = \square$, then also $2(2T - 1)(2T + 1)(T^2 + T - 1)$ is a square. As this product is congruent to 2 modulo 4, we reached a contradiction.

Case 11. Suppose that $2ab(t + 1)$ is a square, equivalently

$$(T + 1)(T + 2)(2T - 1)(T^2 + T - 1)(2T^2 + 3T - 1) = \square.$$

Note that $T^2 + T - 1$ is coprime with $(T + 1)(T + 2)(2T - 1)(2T^2 + 3T - 1)$, so that it must be a perfect square. This implies $(2T + 1)^2 - 5 = \square$, whence $T = 1$ is the only possible value.

Case 12. Assume finally that $2ab(t - 1)$ is a square, that is

$$T(T + 1)(T + 2)(2T + 1)(2T + 3) = \square.$$

Since $\gcd(T(T + 2)(2T + 1)(2T + 3), T + 1) = 1$, one has $T + 1 = \square$ as well as $T(T + 2)(2T + 1)(2T + 3) = \square$. This last equation is readily put into the equivalent form $\left(8(T + 1)^2 - 5\right)^2 - 9 = \square$, which implies that $8(T + 1)^2 = 10$, contradiction. $\square$

In view of Lemmas 5.3 and 5.5, we may apply Theorem 5.4 with $\Gamma = \Lambda'$, that is,

$$b_1 = m,\ b_2 = -n,\ b_3 = 1,$$
$$\alpha_1 = \alpha,\ \alpha_2 = \beta',\quad \alpha_3 = \chi'.$$

Then, $D = 4$ and

$$A_1 = h(\alpha) = \frac{1}{2}\log\alpha,$$
$$A_2 = h(\beta') = \frac{1}{2}\log\beta'.$$

Since all conjugates of $\chi' = (\sqrt{bc} \pm \sqrt{ab})/(\sqrt{ac} \pm \sqrt{ab})$ are greater than one and the minimal polynomial of χ' is

$$a^2(c - b)^2X^4 + 4a^2b(c - b)X^3 + 2ab(3ab - ac - bc - c^2)X^2 + 4ab^2(c - a)X + b^2(c - a)^2$$

divided by the greatest common divisor of the coefficients, which is divisible by $T^2(T+1)^2$, one has

$$A_3 = h(\chi') \leq \frac{1}{4}\log\left(\frac{b^2(c-a)^2}{T^2(T+1)^2}\right) = \frac{1}{2}\log\left(\frac{b(c-a)}{T(T+1)}\right).$$

Assume now that $T > 10^{10}$. Then,

$$\frac{1}{3D}\max\left\{\left|\pm\frac{\log\alpha_1}{A_1} \pm \frac{\log\alpha_2}{A_2} \pm \frac{\log\alpha_3}{A_3}\right|\right\} < 0.35432,$$

which enables us to take $E = 0.35432$. Thus, we may take $C_3^* = 2.8$ and $C_3 = 3$. It is easy to see that C_0 satisfies

$$C_0 \geq \log\left(\frac{C_0 C_2 \omega}{C_1 A_3}\right) = \log(C_0 A)$$

with $A = 34.01472e\, C_1^2 C_2 A_1 A_2$, which allows us to take

$$C_0 = \log A + \log(\log A) + \log(\log(\log A)) + 2\log(\log(\log(\log A)))$$

(note that $\log(\log(\log(\log A))) > 0$ for $T > 10^{10}$). Since $m \geq n$ and $A_1 < A_2$, one has

$$B_0 + B_1 + B_2 + B_3 < \left(\frac{m+1}{4C_0C_2\omega} + \frac{1}{12C_1}\left(\frac{1}{A_1} + \frac{m}{A_3}\right)\right)(1 + \log\beta').$$

On account of $T > 10^{10}$, we may take

$$W_0 = 1 + \log\left(1 + \left(\frac{m+1}{4C_0C_2\omega} + \frac{1}{12C_1}\left(\frac{1}{A_1} + \frac{m}{A_3}\right)\right)(1 + \log\beta')\right).$$

Hence, combining the estimate in Theorem 5.4 with $0 < \Lambda' < (8ac/3)\alpha^{-2m}$ one has

$$m < \frac{91 \cdot 4 \cdot 16 C_0 C_2 W_0 \omega}{\log\alpha} + 1,$$

which together with $m > 3.9999 b^{-1/2} c^{1/2} T^{1/2}$ implies that

$$T < 5.146 \cdot 10^{10}.$$

Now it remains to do the reduction procedure. The required computations ended after 44 days.

Secondly, assume that $c = \gamma_2^-$.

Lemma 5.6. *Let $c = \gamma_2^-$. Assume that $z = v_m = w_n$ holds for some positive integers m and n.*

(i) *If $m \equiv n \equiv 0 \pmod 2$ with $z_0 = z_1 = \varepsilon_1 \in \{\pm 1\}$, then $l \equiv 0 \pmod 2$ and $x_2 = y_2 = 1$.*
(ii) *If $m \equiv n \equiv 1 \pmod 2$ with $z_0 = \varepsilon_2 t$, $z_1 = \varepsilon_2 s$, $\varepsilon_2 \in \{\pm 1\}$, then either $l \equiv 0 \pmod 2$ and $x_2 = T^2 + T - 1$, $y_2 = 2T^3 + 2T^2 - 2T - 1$ or $l \equiv 1 \pmod 2$ and $x_2 = T + 1$, $y_2 = -2T^2 - 2T + 1$.*

Proof. (i) This assertion can be shown by the argument used in the proof of Lemma 4.1.

(ii) The sequence $\{u_n\}$ is given by (18). Since $s \equiv T+1 \pmod 2$ and $t \equiv -2T^2-2T+1 \pmod{2b}$, one has

$$u_n \equiv 2T^3 + 2T^2 - 2T - 1 \pmod{2b}.$$

If l is even, then (12) implies that $2T^3+2T^2-2T-1 \equiv y_2 \pmod{2b}$. Since $|y_2| < \sqrt{b\sqrt{b}}$ by (11), one has

$$y_2 = 2T^3 + 2T^2 - 2T - 1, \quad x_2 = T^2 + T - 1.$$

If l is odd, then (12) shows that

$$2T^3 + 2T^2 - 2T - 1 \equiv bx_2 + ry_2 \pmod{2b}. \tag{40}$$

If $y_2 > 0$, then (18) shows that

$$0 > ry_2 - bx_2 = \frac{y_2^2 - b(b-a)}{ry_2 + bx_2} > -\frac{b^2}{2ry_2} > -b,$$

which contradicts $0 < 2T^3+2T^2-2T-1 < b$ and (40). Hence, $y_2 < 0$. Again from (11) the above congruence is in fact an equality. Combining this equality with $ay_2^2 - bx_2^2 = a - b$, one has

$$x_2^2 + 2a(2T^3 + 2T^2 - 2T - 1)x_2 - (ab + 1 - a^2 + (T^2 - 1)a) = 0.$$

It follows that $x_2 = T + 1$ and $y_2 = -2T^2 - 2T + 1$. $\square$

Remark 5.7. Assume that $y = u_n = u'_l$ for some n and l. Let $u'_{e,\,l} = u'_l$ if l is even with $x_2 = T^2 + T - 1$ and $u'_{o,\,l} = u'_l$ if l is odd with $x_2 = T + 1$. Then, one has $u'_{o,\,0} < u_0 < u'_{e,\,0} = u'_{o,\,1}$ and $u'_{e,\,l} = u'_{o,\,l+1}$ for all even positive integers l. Therefore, it suffices to consider the case where l is even.

Let us first consider case (B). In view of Remark 5.7, we may assume that l is even. By the standard technique, one can show the inequalities

$$0 < \Gamma := l \log \beta - n \log \beta' + \log \chi'' < (\beta')^{2-2n}, \tag{41}$$

where

$$\beta = r + \sqrt{ab}, \quad \beta' = t + \sqrt{bc}, \quad \chi'' = \frac{\sqrt{c}(y_2\sqrt{a} + x_2\sqrt{b})}{\sqrt{a}(r\sqrt{c} + \varepsilon_2 s\sqrt{b})},$$
$$x_2 = T^2 + T - 1, \quad y_2 = 2T^3 + 2T^2 - 2T - 1.$$

Lemma 5.8. *Assume that $z = v_m = w_n$ holds with $m \equiv n \equiv 1 \pmod 2$ and l even. Put $\Delta' = n - l/2$. If $n > 1$, then $\Delta' \geq 1$.*

Proof. We show by induction that $u_n < u'_{2n}$ for $n \geq 1$ if $\tau = -$ and for $n > 1$ if $\tau = +$, since in the latter case one has $u_1 = u'_2$, which corresponds to $d = d_+$.

In the case of $\tau = -$, one has $u_1 = u'_0 < u'_2$. For $n \geq 2$, the inductive assumption together with $t < 2r^2 - 2$ implies that

$$\begin{aligned} u_n &< 2tu_{n-1} < 2(2r^2-2)u'_{2n-2} = 2r(u'_{2n-1} + u'_{2n-3}) - 4u'_{2n-2} \\ &= u'_{2n} + u'_{2n-4} - 2u'_{2n-2} < u'_{2n}. \end{aligned}$$

In the case of $\tau = +$, one easily sees that one has $u_2 < u'_4$. For $n \geq 3$, one obtains $u_n < u'_{2n}$ in a similar way to the previous case. □

Lemma 5.9. *Assume that $v_m = w_n$ holds with $m \equiv n \equiv 1 \pmod 2$ and l even. Put $\Delta' = n - l/2$. If $T > 1000$, then*

$$n > \frac{2\Delta' - 1}{2.0002} \cdot T \log \beta'.$$

Proof. Noting that

$$\begin{aligned} 0 < \beta^2 - \beta' = (r + \sqrt{ab})^2 - (t + \sqrt{bc}) &< 4ab + 3 - 2t \\ &= 16T^5 + 48T^4 + 32T^3 - 12T^2 - 12T + 1, \\ \beta' > 2(t-1) &= 16T^6 + 48T^5 + 16T^4 - 48T^3 - 20T^2 + 12T \end{aligned}$$

and $T > 1000$, one has

$$\frac{\beta^2 - \beta'}{\beta'} < \frac{16T^5 + 48T^4 + 32T^3 - 12T^2 - 12T + 1}{16T^6 + 48T^5 + 16T^4 - 48T^3 - 20T^2 + 12T} < \frac{1.0001}{T}.$$

Since the inequality $\Gamma > 0$ implies $(l + 1) \log \beta > n \log \beta'$, one obtains

$$\frac{2\Delta' - 1}{n} = 2 - \frac{l+1}{n} < 2 - \frac{\log \beta'}{\log \beta} < \frac{\beta^2 - \beta'}{\beta' \log \beta} < \frac{1.0001}{T \log \beta} < \frac{2.0002}{T \log \beta'},$$

from which the assertion immediately follows. $\square$

Proposition 5.10. *Let $a = T^2 + 2T$, $b = 4T^4 + 8T^3 - 4T$ and $c = 16T^8 + 64T^7 + 48T^6 - 80T^5 - 88T^4 + 32T^3 + 36T^2 - 4T - 3$. Assume that $z = v_m = w_n$ holds for some odd positive integers m and n with $z_0 = \varepsilon_2 t$, $z_1 = \varepsilon_2 s$ and $\varepsilon_2 \in \{\pm 1\}$, and l even. Then, $T \le 6585$.*

Proof. The linear form Γ can be rewritten as

$$\Gamma = \frac{l}{2} \log\left(\frac{\beta^2}{\beta'}\right) - \log\left(\frac{(\beta')^{\Delta'}}{\chi''}\right).$$

We apply Proposition 4.2 with $b_1 = 1,\ b_2 = l/2,\ \alpha_1 = (\beta')^{\Delta'}/\chi'',\ \alpha_2 = \beta^2/\beta'$. Since the conjugates of α_2 greater than one are

$$\frac{(r + \sqrt{ab})^2}{t + \sqrt{bc}}, \quad \frac{(r + \sqrt{ab})^2}{t - \sqrt{bc}},$$

one has

$$h(\alpha_2) = \frac{1}{4} \log(r + \sqrt{ab})^4 = \log \beta.$$

Since the conjugates of χ'' greater than one are

$$\frac{\sqrt{c}(y_2\sqrt{a} + x_2\sqrt{b})}{\sqrt{a}(r\sqrt{c} - s\sqrt{b})}, \quad \frac{\sqrt{c}(-y_2\sqrt{a} + x_2\sqrt{b})}{\sqrt{a}(r\sqrt{c} - s\sqrt{b})},$$

one has

$$\begin{aligned} h(\chi'') &\le \frac{1}{4} \log\left(a^2(c-b)^2 \cdot \frac{c}{a} \cdot \frac{b-a}{(r\sqrt{c} - s\sqrt{b})^2}\right) \\ &< \frac{1}{4} \log\left(4(ab+1)ac^2(b-a)\right) < \frac{1}{2}\log(2abc) \\ &< \frac{7}{12}\log(2bc) < \frac{7}{6}\log \beta', \end{aligned}$$

where the second to last inequality follows from $a^6 < bc$. Hence, one obtains

$$h(\alpha_1) \le \Delta' h(\beta') + h(\chi'') < \frac{3\Delta' + 7}{6} \log \beta'.$$

In what follows, assume that $T > 3000$. Putting $\rho = 37$ and $\mu = 0.63$, one may take a_1 and a_2 as

$$a_1 = \left(40\varDelta' + \frac{28}{3}\right) \log \beta',$$
$$a_2 = 4 \log \beta' + 0.0136,$$

which together imply that

$$\frac{b_1}{a_2} + \frac{b_2}{a_1} = \frac{1}{4 \log \beta' + 0.0136} + \frac{l/2}{(40\varDelta' + 28/3) \log \beta'}$$
$$< \frac{l/2 + 10\varDelta' + 7/3}{(40\varDelta' + 28/3) \log \beta'}.$$

Thus, one may take

$$h = 4 \log \frac{n + 10\varDelta' + 7/3}{(40\varDelta' + 28/3) \log \beta'} + 11.913.$$

If $h \leq 28.71$, then $n < 66.637(40\varDelta' + 28/3) \log \beta'$. If $h > 28.71$, then Proposition 4.2 and $0 < \Gamma < (\beta')^{2-2n}$ together show that $n < 66.734(40\varDelta' + 28/3) \log \beta'$, which holds in any case. Combining this inequality with the one in Lemma 5.9, one obtains

$$T < 2.0002 \cdot 66.734 \cdot \frac{40\varDelta' + 28/3}{2\varDelta' - 1} < 6585.1,$$

that is, $T \leq 6585$. $\square$

Using the same methods, we can get a similar result in the case (A), only some constants will be different, so we do not give all details here. More precisely, we can define the same linear form in logarithms Γ, but with $z_0 = z_1 = \pm 1$ and $x_2 = y_2 = 1$ which will change the number χ''. Here, we can also prove $0 < \Gamma < (\beta')^{2-2n}$. As we said above, in this case l is even, and in the same way as in the case (B) we have $\varDelta' = n - l/2 \geq 1$. As in the proof of Lemma 5.9, assuming $T > 3000$, we get that $v_m = w_n$, with $m \equiv n \equiv 0 \pmod 2$, implies

$$n > \frac{2\varDelta' - 1}{2.0002} \cdot T \log \beta'.$$

Furthermore, applying Proposition 4.2, the only difference from the case (B) is that

$$h(\chi'') < \log \beta'.$$

Moreover, we can take

$$a_1 = (40\varDelta' + 8) \log \beta',$$
$$a_2 = 4 \log \beta' + 0.0134.$$

Defining

$$h = 4\log\frac{n+10\Delta'+2}{(40\Delta'+8)\log\beta'} + 11.913,$$

we can get

$$n < 66.743(40\Delta'+8)\log\beta',$$

which at the end gives us $T \le 6407$.

Explicit computations required by the reduction method give in less than three minutes $n \le 7$, in contradiction with the inequality

$$n > \frac{2\Delta'-1}{2.052}\cdot T\log\beta'$$

obtained by the same argument as in the proof of Lemma 5.9 with $T \ge 6$. This shows that Theorem 1.1 holds when $c = \gamma_2^-$.

Thirdly, consider the case where $c = \gamma_2^+$.

Suppose that $T \ge 100$. Since $b < 4a^2$, $c > 4b^3$ and $\gcd(a,b) \ge T \ge 100$, it holds $3.804b^2(b-a)^3 < abcT^4$. One can thus apply [8, Theorem 2.1], which together with [12, Lemma 12] and $\log z > (n/2)\log(4bc)$ (cf. [19, Lemma 25]) shows a similar inequality to the one in [8, Lemma 3.2]:

$$n < \frac{8\log(8.4706\cdot 10^{13}a^{1/2}(b-a)^{1/2}b^2cT^{-1})\log(1.6215a^{1/2}b^{1/2}(b-a)^{-1}cT)}{\log(4bc)\log(0.2629ab^{-1}(b-a)^{-3}cT^4)}.$$

On account of $T \ge 100$, $a \ge 10200$, $b > 3a^2 \ge 3.12\cdot 10^8$ and

$$\begin{aligned} a^{1/2}T^{-1} &= \left(1+\frac{2}{T}\right)^{1/2} < 1.01,\\ \frac{a^{1/2}b^{1/2}T}{b-a} &= \frac{T(T^2+2T)^{1/2}}{b^{1/2}(1-a/b)} < 0.501,\\ \frac{aT^4}{b(b-a)^3} &> \frac{aT^4}{b^4} > \frac{1}{8.08b^{5/2}}, \end{aligned}$$

we have

$$n < \frac{8\log(8.556\cdot 10^{13}b^{5/2}c)\log(0.8124c)}{\log(4bc)\log(0.03253b^{-5/2}c)}. \tag{42}$$

Since the right-hand side is a decreasing function of c and $c > 4b^3$, we see that

$$n < \frac{8\log(3.4224\cdot 10^{14}b^{11/2})\log(3.2496b^3)}{\log(16b^4)\log(0.13012b^{1/2})}. \tag{43}$$

(A) In the case where $m \equiv n \equiv 0 \pmod 2$, we deduce from the proof of [7, Lemma 2.4] that $m > b^{-1/2}c^{1/2}$. Since the inequality $m \le (4n+2)/3$ for $c > 4b^3$ can be deduced in the same way as [13, Lemma 4], it holds

$$n \ge \frac{3m-2}{4} > \frac{3}{4}b^{-1/2}c^{1/2} - \frac{1}{2} > 1.5b - 0.5. \tag{44}$$

Inequalities (43) and (44) together imply that $b < 303$, which is a contradiction.

(B) In the case where $m \equiv n \equiv 1 \pmod 2$, as in the proof of [20, Lemma 3.1 (2)], we have

$$\pm t\left\{a(m^2-1) - b(n^2-1)\right\} \equiv 2rs(n-m) \pmod{8c}, \tag{45}$$

$$\pm s\left\{a(m^2-1) - b(n^2-1)\right\} \equiv 2rt(n-m) \pmod{8c}. \tag{46}$$

Since $c = \gamma_2^+$ and

$$s = 16T^7 + 80T^6 + 124T^5 + 36T^4 - 52T^3 - 22T^2 + 5T + 1,$$
$$t = 32T^8 + 160T^7 + 248T^6 + 56T^5 - 152T^4 - 72T^3 + 26T^2 + 10T - 1,$$

we have $2tb = c - A$, where

$$A := 256T^{11} + 1728T^{10} + 4288T^9 + 4176T^8 - 256T^7 - 3072T^6 - 1152T^5 + 672T^4 + 320T^3 - 76T^2 - 20T + 5.$$

Thus from (45) we deduce

$$\pm\left\{2ta(m^2-1) + A(n^2-1)\right\} \equiv 4rs(n-m) \pmod{c}. \tag{47}$$

Suppose now that $\max\{m, n^2\} \le T/2$. Then, since

$$2tam^2 + An^2 \le \frac{T^2ta}{2} + \frac{AT}{2} < \frac{3c}{4},\ 4rsm \le 2Trs < \frac{c}{4},$$
$$sam^2 \le \frac{T^2sa}{4} < \frac{c}{4},\ sbn^2 \le \frac{Tsb}{2} < \frac{c}{2},\ 2rtm \le Trt < \frac{c}{4},$$

congruences (46) and (47) are in fact equalities. It follows that

$$2a(b-a)(m^2-1) = (2b^2 - 2ab - t)(n^2-1). \tag{48}$$

The left-hand side of (48) is divisible by T^2 (because $a = T^2 + 2T$), while $2b^2 - 2ab - t \equiv 1 \pmod T$, which implies that $n^2 - 1 \equiv 0 \pmod{T^2}$. Since $n > 1$, one obtains the contradiction $n^2 \ge T^2 - 1 > (T+1)/2$.

We conclude that $\max\{m, n^2\} > T/2$. If $n^2 > T/2$, then $n^2 \geq (T+1)/2$. If $m > T/2$, since $m \leq (4n+2)/3$ as seen above, it holds $5n/3 \geq m > T/2$, which together with $T \geq 100$ gives $n^2 \geq (T+1)/2$.

Hence, one always has $n^2 \geq (T+1)/2$, which combined with (43) shows that $T \leq 14190$. Therefore, the reduction method easily leads us to a contradiction. The computation time was in this case less than five minutes.

Finally, assume that $c \geq \gamma_3^-$. Suppose that $T \geq 100$. In the same way as in the previous case, one has inequality (42). One sees from $c > b^{7/2}$ that

$$n < \frac{8\log(8.556 \cdot 10^{13} b^6)\log(0.8124 b^{7/2})}{\log(4b^{9/2})\log(0.03253b)}. \tag{49}$$

(A) In the case where $m \equiv n \equiv 0 \pmod 2$, we deduce from the proof of [7, Lemma 2.4] that $m > b^{-1/2}c^{1/2}$. Since $m \leq (9n+5)/7$ by [20, Lemma 2.4] with $c > b^{7/2}$, and $n \geq 4$ by [20, Lemma 2.5], it holds

$$n > 0.6829 b^{-1/2} c^{1/2} > 0.6829 b^{5/4}. \tag{50}$$

Inequalities (49) and (50) together imply that $b < 117$, which is a contradiction.

(B) In the case where $m \equiv n \equiv 1 \pmod 2$, as in the case where $c = \gamma_2^+$, we have congruences (45) and (46).

Suppose that $n \leq 2.826 b^{-3/4} c^{1/4}$. By [13, Lemma 3], [20, Lemmas 2.4 and 2.5] and $c > b^{7/2}$, we have

$$n \leq m \leq \frac{9n+5}{7} \leq \frac{10n}{7}.$$

Since

$$t\left|a(m^2-1) - b(n^2-1)\right| < btn^2 < 7.99c,$$
$$2rt(m-n) < 2rt \cdot \frac{3}{7}n < 0.01c,$$

congruences (45) and (46) are in fact equalities. It immediately results that $m = n$ and $a(m^2-1) = b(n^2-1)$, which contradict $a < b$. Hence we have

$$n > 2.826 b^{-3/4} c^{1/4} > 2.826 b^{1/8}. \tag{51}$$

It follows from inequalities (49) and (51) that $b < 1.502 \cdot 10^{10}$. Since $b = 4T^4 + 8T^3 - 4T$, we obtain $T \leq 247$. Therefore, the reduction method easily leads us to a contradiction. The computation time was in this case less than three minutes.

This completes the proof of Theorem 1.3 and hence of Theorem 1.1.

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