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Generalizations of Hardy-Type Inequalities by Montgomery Identity and New Green Functions

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Abstract: In this paper we extend general Hardy's inequality by appropriately combining Montgomery's identity and Green functions. Related Grüss and Ostrowski-type inequalities are also derived.

Keywords: n-convexity; Montgomery identity; Hardy inequality; Grüss-type inequality; Ostrowsky-type inequality

MSC: 26D10, 26D15

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1. Introduction

The area of mathematical inequalities is very large. There are many mathematicians that study, improve, and generalize many inequalities such as the Hardy, the Hardy–Hilbert, the Steffensen, the Opial, the Boas, etc. Here, we focus on the famous Hardy's inequality; see [1]. It states that

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)\,dt \right)^p dx \le \left(\frac{p}{p-1} \right)^p \int_0^\infty f^p(x)\,dx,\ p > 1, \tag{1}$$

holds for all non-negative functions $f \in L^p(0,\infty)$, with the constant $\left(\frac{p}{p-1}\right)^p$ being sharp. Inequality (1) has important applications in operator theory since it can be reinterpreted as

$$\|Hf\|_p \le \frac{p}{p-1}\|f\|_p$$

where $Hf(x) = \frac{1}{x}\int_0^x f(t)\,dt$ is the Hardy operator and $\|\cdot\|_p$ is the standard L^p norm. For these reasons, Hardy's inequality, both in discrete and continuous case, has attracted a lot of interest of researchers, and there are many papers and monographs dedicated to its development; here, we mention, e.g., [2–13].

We continue with Pólya–Knopp's inequality,

$$\int_0^\infty \exp\left(\frac{1}{x} \int_0^x \ln f(t)\,dt \right) dx < e \int_0^\infty f(x)\,dx, \tag{2}$$

which holds for positive functions $f \in L^1(\mathbb{R}_+)$. Pólya–Knopp's inequality is a limiting case of Hardy's inequality, since it can can be obtained from (1) by rewriting it with the function f replaced with $f^{\frac{1}{p}}$ and then by letting $p \to \infty$.

The starting point of our paper will be Hardy's inequality in the general setting given in [14] (see also [15]), and we will first introduce some notation. Let $(\Sigma_i, \Omega_i, \mu_i)$, $i = 1, 2$, be measure spaces with positive σ-finite measures, $k : \Omega_1 \times \Omega_2 \to [0, \infty)$ a measurable and non-negative kernel and

$$0 < K(x) = \int_{\Omega_2} k(x,t) d\mu_2(t), \qquad x \in \Omega_1. \tag{3}$$

For a measurable $f : \Omega_2 \to \mathbb{R}$, let A_k denote the integral operator

$$A_k f(x) = \frac{1}{K(x)} \int_{\Omega_2} k(x,t) f(t) d\mu_2(t), \tag{4}$$

Theorem 1 ([14])**.** *Let the weight $u : \Omega_1 \to [0, \infty)$ and kernel $k : \Omega_1 \times \Omega_2 \to [0, \infty)$ be such that $\frac{k(x,y)}{K(x)} u(x)$ is locally integrable on Ω_1 for each $y \in \Omega_2$ and let v be given by*

$$v(y) = \int_{\Omega_1} \frac{k(x,y)}{K(x)} u(x) d\mu_1(x) < \infty. \tag{5}$$

If ϕ is a convex function on an interval $I \subseteq \mathbb{R}$, then the inequality

$$\int_{\Omega_1} \phi(A_k f(x)) u(x) d\mu_1(x) \le \int_{\Omega_2} \phi(f(y)) v(y) d\mu_2(y) \tag{6}$$

holds for all measurable functions $f : \Omega_2 \to I$, where A_k is given by (4).

Inequality (6) is, indeed, a generalization of Hardy's inequality. After taking $f(t) = g(t^{\frac{p-1}{p}}) t^{\frac{-1}{p}}$ and some straightforward calculation, we obtain that (1) is equivalent to

$$\int_0^\infty \left(\frac{1}{x} \int_0^x g(t)\, dt \right)^p \frac{dx}{x} \le \int_0^\infty g^p(x)\, \frac{dx}{x}. \tag{7}$$

Applying (6) with $\Omega_1 = \Omega_2 = (0, \infty)$, $k(x,y) = 1$ for $0 \le y \le x$ and $k(x,y) = 0$ otherwise, $d\mu_1(x) = d\mu_2(x) = dx$, $u(x) = \frac{1}{x}$ (which yields $v(y) = \frac{1}{y}$) and $\phi(u) = u^p$ gives (7).

We will state our results for the class of n-convex functions, which is a more general class of functions that contains convex functions as a special case. We will now recall some basic definitions and properties of n-convex functions.

Definition 1. *The n-th order divided difference, $n \in \mathbb{N}_0$, of a function $\phi : [\alpha, \beta] \to \mathbb{R}$ at mutually distinct points $x_0, x_1, \dots, x_n \in [\alpha, \beta]$ is defined recursively by*

$$[x_i; \phi] = \phi(x_i), \quad i = 0, \dots, n$$
$$[x_0, \dots, x_n; \phi] = \frac{[x_1, \dots, x_n; \phi] - [x_0, \dots, x_{n-1}; \phi]}{x_n - x_0}.$$

The value $[x_0, \dots, x_n; \phi]$ is independent of the order of the points $x_0, \dots, x_n$. A function $f : [\alpha, \beta] \to \mathbb{R}$ is n-convex if all its n-th order divided differences are non-negative, i.e., $[x_0, \dots, x_n; f] \ge 0$ for all choices $x_i \in [\alpha, \beta]$. Thus, 0-convex functions are non-negative and 1-convex functions are non-decreasing, while 2-convex functions are convex in the classical sense. An n times differentiable is n-convex if and only if its n-derivative is non-negative (see [10]).

In our presentation, we will also need the following generalization of the Montgomery identity given in [16].

Theorem 2. *Let $n \in \mathbb{N}$, $\phi : I \to \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous, where I is an open interval in $\mathbb{R}$ and $\alpha, \beta \in I$, $\alpha < \beta$. Then*

$$\phi(t) = \frac{1}{\beta-\alpha}\int_\alpha^\beta \phi(s)ds + \sum_{k=0}^{n-2} \frac{\phi^{(k+1)}(\alpha)}{k!(k+2)}\frac{(t-\alpha)^{k+2}}{\beta-\alpha} - \sum_{k=0}^{n-2} \frac{\phi^{(k+1)}(\beta)}{k!(k+2)}\frac{(t-\beta)^{k+2}}{\beta-\alpha} + \frac{1}{(n-1)!}\int_\alpha^\beta T_n(t,s)\phi^{(n)}(s)ds, \tag{8}$$

where

$$T_n(t,s) = \begin{cases} -\frac{(t-s)^n}{n(\beta-\alpha)} + \frac{t-\alpha}{\beta-\alpha}(t-s)^{n-1}, & \alpha \le s \le t, \\ -\frac{(t-s)^n}{n(\beta-\alpha)} + \frac{t-\beta}{\beta-\alpha}(t-s)^{n-1}, & t < s \le \beta. \end{cases} \tag{9}$$

For $n = 1$, the sums $\sum_{k=0}^{n-2}$ are empty and identity (8) reduces to the well-known Montgomery identity

$$\phi(t) = \frac{1}{\beta-\alpha}\int_\alpha^\beta \phi(s)ds + \int_\alpha^\beta P(t,s)\phi'(s)ds,$$

where $P(t,s)$ is the Peano kernel, which is defined by

$$P(t,s) = \begin{cases} \frac{s-\alpha}{\beta-\alpha}, & \alpha \le s \le t, \\ \frac{s-\beta}{\beta-\alpha}, & t < s \le \beta. \end{cases}$$

Hardy-type inequalities obtained by similar methods as in this paper were given in [15,17–20]. We also draw attention to two papers [21,22] about Sherman's inequality and Montgomery identity.

2. Main Results

Throughout the paper, G_ω, $\omega = 1,2,3,4$, will denote the following Green functions defined on $[\alpha,\beta] \times [\alpha,\beta]$ with

$$G_1(t,s) = \begin{cases} \alpha - s, & \alpha \le s \le t; \\ \alpha - t, & t \le s \le \beta. \end{cases} \tag{10}$$

$$G_2(t,s) = \begin{cases} t - \beta, & \alpha \le s \le t; \\ s - \beta, & t \le s \le \beta. \end{cases} \tag{11}$$

$$G_3(t,s) = \begin{cases} t - \alpha, & \alpha \le s \le t; \\ s - \alpha, & t \le s \le \beta. \end{cases} \tag{12}$$

$$G_4(t,s) = \begin{cases} \beta - s, & \alpha \le s \le t; \\ \beta - t, & t \le s \le \beta. \end{cases} \tag{13}$$

Note that all these functions are continuous and convex with respect to the first variable.

Lemma 1. *For $\phi \in C^2([\alpha,\beta])$, the following identities hold*

$$\phi(t) = \phi(\alpha) + (t-\alpha)\phi'(\beta) + \int_\alpha^\beta G_1(t,s)\phi''(s)ds, \tag{14}$$

$$\phi(t) = \phi(\beta) + (t-\beta)\phi'(\alpha) + \int_\alpha^\beta G_2(t,s)\phi''(s)ds, \tag{15}$$

$$\phi(t) = \phi(\beta) + (t-\alpha)\phi'(\alpha) - (\beta-\alpha)\phi'(\beta) + \int_\alpha^\beta G_3(t,s)\phi''(s)ds, \tag{16}$$

$$\phi(t) = \phi(\alpha) - (\beta-t)\phi'(\beta) + (\beta-\alpha)\phi'(\alpha) + \int_\alpha^\beta G_4(t,s)\phi''(s)ds, \tag{17}$$

where the functions $G_\omega,\ \omega = 1,\dots,4$, are defined by (10)–(13).

The next theorem gives our first main result.

Theorem 3. *Let A_k be as in (4), $\omega \in \{1,2,3,4\}$, G_ω as in (10)–(13) and u a weight function with v given by (5). Then, the following statements are equivalent:*

(i) For every continuous convex function $\phi : [\alpha,\beta] \to \mathbb{R}$, we have

$$\int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) \le \int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) \tag{18}$$

(ii) For each $s \in [\alpha,\beta]$, we have

$$\int_{\Omega_1} G_\omega(A_k f(x),s)u(x)d\mu_1(x) \le \int_{\Omega_2} G_\omega(f(y),s)v(y)d\mu_2(y) \tag{19}$$

Proof. (i) $\Rightarrow$ (ii): The functions $G_\omega(\cdot,s)$, $s \in [\alpha,\beta]$. $\omega = 1,2,3,4$, are continuous and convex, and applying these functions to (18), we obtain (19).

(ii) $\Rightarrow$ (i): Let $\phi \in C^2([\alpha,\beta])$. Identities (14)–(17) and some simple calculations yield

$$\begin{aligned}
&\int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) \\
&= \int_\alpha^\beta \left[\int_{\Omega_2} G_\omega(f(y),s)v(y)d\mu_2(y) - \int_{\Omega_1} G_\omega(A_k f(x),s)u(x)d\mu_1(x) \right] \phi''(s)ds
\end{aligned} \tag{20}$$

for $\omega = 1,2,3,4$. If, additionally, ϕ is convex, then $\phi''(s) \ge 0$ for $s \in [\alpha,\beta]$. Furthermore, by assumption (19), the first factor under the integral on the right-hand side is also non-negative. Therefore, the right-hand side of (20) is non-negative, so the left-hand side is as well, i.e., (18) holds. Since each continuous convex function on a segment can be attained as a uniform limit of C^2 convex functions, the claim of the theorem follows. □

In the next theorem, we will use the Montgomery identity to obtain general identities. In turn, these identities will be used in derivation of Hardy-type inequalities.

Theorem 4. *Let $n \in \mathbb{N}$, $n \ge 4$ and $\phi : I \to \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous, where I is an open interval in $\mathbb{R}$ and $\alpha, \beta \in I$, $\alpha < \beta$. Furthermore, let A_k be as in (4), $\omega \in \{1,2,3,4\}$, G_ω as in (10)–(13), T_n as in (9) and u a weight function with v given by (5). Then*

(i)

$$\int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x)$$
$$= \int_\alpha^\beta \left[\int_{\Omega_2} G_\omega(f(y),t)v(y)d\mu_2(y) - \int_{\Omega_1} G_\omega(A_k f(x),t)u(x)d\mu_1(x) \right]$$
$$\times \sum_{k=1}^{n-1} \frac{k}{(k-1)!} \left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1} - \phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta-\alpha} \right) dt$$
$$+ \frac{1}{(n-3)!} \int_\alpha^\beta \int_\alpha^\beta \left[\int_{\Omega_2} G_\omega(f(y),t)v(y)d\mu_2(y) - \int_{\Omega_1} G_\omega(A_k f(x),t)u(x)d\mu_1(x) \right] \cdot$$
$$\cdot \tilde{T}_{n-2}(t,s)\phi^{(n)}(s)dsdt, \tag{21}$$

where

$$\tilde{T}_{n-2}(t,s) = \begin{cases} \frac{1}{\beta-\alpha}\left[\frac{(t-s)^{n-2}}{n-2} + (t-\alpha)(t-s)^{n-3}\right], & \alpha \le s \le t, \\ \frac{1}{\beta-\alpha}\left[\frac{(t-s)^{n-2}}{n-2} + (t-\beta)(t-s)^{n-3}\right], & t < s \le \beta. \end{cases} \tag{22}$$

(ii)

$$\int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x)$$
$$= \frac{1}{2}\frac{\phi'(\beta)-\phi'(\alpha)}{\beta-\alpha} \left[\int_{\Omega_2} v(y)f^2(y)d\mu_2(y) - \int_{\Omega_1} u(x)A_k^2 f(x)d\mu_1(x) \right]$$
$$+ \int_\alpha^\beta \left[\int_{\Omega_2} G_\omega(f(y),t)v(y)d\mu_2(y) - \int_{\Omega_1} G_\omega(A_k f(x),t)u(x)d\mu_1(x) \right] \times$$
$$\sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1} - \phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta-\alpha} \right) dt$$
$$+ \frac{1}{(n-3)!} \int_\alpha^\beta \int_\alpha^\beta \left[\int_{\Omega_2} G_\omega(f(y),t)v(y)d\mu_2(y) - \int_{\Omega_1} G_\omega(A_k f(x),t)u(x)d\mu_1(x) \right] \cdot$$
$$\cdot T_{n-2}(t,s)\phi^{(n)}(s)dsdt. \tag{23}$$

Proof.

(i) If we differentiate twice the Montgomery identity (8), we obtain

$$\phi''(t) = \sum_{k=1}^{n-1} \frac{k}{(k-1)!} \left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1} - \phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta-\alpha} \right)$$
$$+ \frac{1}{(n-3)!} \int_\alpha^\beta \tilde{T}_{n-2}(t,s)\phi^{(n)}(s)ds. \tag{24}$$

Inserting (24) in (20), we derive the first identity (21).

(ii) Substituting ϕ with ϕ'' and n with $n-2$ in (8) gives

$$\phi''(t) = \frac{1}{\beta-\alpha}\int_\alpha^\beta \phi''(t)dt + \sum_{k=0}^{n-4}\frac{\phi^{(k+3)}(\alpha)}{k!(k+2)}\frac{(t-\alpha)^{k+2}}{\beta-\alpha} - \sum_{k=0}^{n-4}\frac{\phi^{(k+3)}(\beta)}{k!(k+2)}\frac{(t-\beta)^{k+2}}{\beta-\alpha}$$
$$+\frac{1}{(n-3)!}\int_\alpha^\beta T_{n-2}(t,s)\phi^{(n)}(s)ds,$$

so

$$\phi''(t) = \frac{\phi'(\beta)-\phi'(\alpha)}{\beta-\alpha} + \sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1}-\phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta-\alpha}\right)$$
$$+\frac{1}{(n-3)!}\int_\alpha^\beta T_{n-2}(t,s)\phi^{(n)}(s)ds. \tag{25}$$

Furthermore, Lemma 1 applied for $\phi(t)=\frac{1}{2}t^2$ shows that

$$\int_\alpha^\beta G_\omega(z,t)dt = \frac{1}{2}z^2 + P(z,\alpha,\beta)$$

for $\omega=1,2,3,4$, where P is a linear polynomial in the variable z. This fact, after some straightforward calculation, yields

$$\int_\alpha^\beta \left[\int_{\Omega_2} G_\omega(f(y),t)v(y)d\mu_2(y) - \int_{\Omega_1} G_\omega(A_kf(x),t)u(x)d\mu_1(x)\right]dt =$$
$$\int_{\Omega_2} v(y)\frac{f^2(y)}{2}d\mu_2(y) - \int_{\Omega_1} u(x)\frac{A_k^2f(x)}{2}d\mu_1(x) \tag{26}$$

for $\omega=1,2,3,4$. Finally, inserting (25) in (20) and taking into account (26) gives (23).
□

In the following theorem, new Hardy-type inequalities are derived from the above identities.

Theorem 5. *Suppose that all the assumptions of Theorem 4 hold with the additional assumption that n is even. If $\phi : I \to \mathbb{R}$ is n-convex, then:*

(i)

$$\int_{\Omega_2}\phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1}\phi(A_kf(x))u(x)d\mu_1(x)$$
$$\geq \int_\alpha^\beta \left[\int_{\Omega_2} G_\omega(f(y),t)v(y)d\mu_2(y) - \int_{\Omega_1} G_\omega(A_kf(x),t)u(x)d\mu_1(x)\right]$$
$$\times\sum_{k=1}^{n-1}\frac{k}{(k-1)!}\left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1}-\phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta-\alpha}\right)dt, \tag{27}$$

(ii)

$$\begin{aligned}&\int\limits_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int\limits_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x)\\ &\geq \frac{1}{2}\frac{\phi'(\beta)-\phi'(\alpha)}{\beta-\alpha}\left[\int\limits_{\Omega_2} v(y)f^2(y)d\mu_2(y) - \int\limits_{\Omega_1} u(x)A_k^2 f(x)d\mu_1(x)\right]\\ &+\int_\alpha^\beta\left[\int\limits_{\Omega_2} G_\omega(f(y),t)v(y)d\mu_2(y) - \int\limits_{\Omega_1} G_\omega(A_k f(x),t)u(x)d\mu_1(x)\right]\times\\ &\sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1}-\phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta-\alpha}\right)dt. \end{aligned} \tag{28}$$

Proof. Since for each $\omega \in \{1,2,3,4\}$ and $t \in [\alpha,\beta]$ the function $G_\omega(\cdot,s)$ is continuous and convex, Theorem 1 yields

$$\int\limits_{\Omega_2} G_\omega(f(y),t)v(y)d\mu_2(y) - \int\limits_{\Omega_1} G_\omega(A_k f(x),t)u(x)d\mu_1(x) \geq 0, \quad \text{for all} \quad t \in [\alpha,\beta]. \tag{29}$$

Furthermore, for even n, the function $\tilde{T}_{n-2}$ given by (22) is obviously non-negative, while the function T_{n-2} is also non-negative since

$$T_{n-2}(t,s) = \begin{cases} \frac{(t-s)^{n-3}}{(\beta-\alpha)}\left[t-a-\frac{t-s}{n}\right], & \alpha \leq s \leq t,\\ \\ \frac{(s-t)^{n-3}}{(\beta-\alpha)}\left[\beta-t-\frac{s-t}{n}\right], & t < s \leq \beta. \end{cases}$$

Furthermore, $\phi^{(n)} \geq 0$ since ϕ is n-convex. Finally, from identities (21) and (23), due to the positivity of $\phi^{(n)}$, $\tilde{T}_{n-2}$ and T_{n-2} and inequality (29), we obtain inequalities (27) and (28), respectively. □

With additional convexity assumptions, the right-hand sides of inequalities (27) and (28) can be further simplified.

Theorem 6. *Suppose that all the assumptions of Theorem 5 hold and denote the functions*

$$L_1(z) = \int_\alpha^\beta G_w(z,t) \times \sum_{k=1}^{n-1}\frac{k}{(k-1)!}\left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1}-\phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta-\alpha}\right)dt \tag{30}$$

and

$$\begin{aligned}L_2(z) = \frac{\phi'(\beta)-\phi'(\alpha)}{\beta-\alpha}\frac{z^2}{2} + \int_\alpha^\beta G_w(z,t)\times\\ \sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1}-\phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta-\alpha}\right)dt.\end{aligned} \tag{31}$$

If ϕ is n-convex and L_1 or L_2 are convex, then

$$\int\limits_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) \leq \int\limits_{\Omega_2} \phi(f(y))v(y)d\mu_2(y).$$

Proof. The right-hand side of inequality (27) can be rewritten as

$$\int_{\Omega_2} v(y) L_1(f(y)) d\mu_2(y) - \int_{\Omega_1} u(x) L_1(A_k f(x)) d\mu_1(x).$$

Since L_1 is convex, by Theorem 1, the last expression is non-negative. Therefore, the stated inequality follows from (27).

The claim for the function L_2 follows from inequality (28) in an analogous way. □

Remark 1. *Differentiating twice the identities from Lemma 1, we can conclude*

$$g''(z) = \frac{\partial^2}{\partial z^2} \int_\alpha^\beta G_\omega(z,t) g''(t) dt. \tag{32}$$

Applying (32) for a function g such that

$$g''(t) = \sum_{k=1}^{n-1} \frac{k}{(k-1)!} \left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1} - \phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta - \alpha} \right) \tag{33}$$

we can conclude

$$L_1''(z) = \frac{\partial^2}{\partial z^2} \int_\alpha^\beta G_\omega(z,t) g''(t) dt = g''(z).$$

Therefore, the convexity of the function L_1 is equivalent to the non-negativity of the expression (33) for each $t \in [\alpha, \beta]$.

Similarly, taking also into account (26), one can show that the convexity of the function L_2 is equivalent to

$$\frac{\phi'(\beta) - \phi'(\alpha)}{\beta - \alpha} + \sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1} - \phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta - \alpha} \right) \geq 0$$

for each $t \in [\alpha, \beta]$.

3. Related Grüss and Ostrowski-Type Inequalities

Consider the Čebyšev functional

$$T(h,g) := \frac{1}{\beta - \alpha} \int_\alpha^\beta h(t) g(t) dt - \frac{1}{\beta - \alpha} \int_\alpha^\beta h(t) dt \cdot \frac{1}{\beta - \alpha} \int_\alpha^\beta g(t) dt$$

for Lebesgue integrable functions $h, g : [\alpha, \beta] \to \mathbb{R}$. The next two theorems from [23] provide Grüss and Ostrowski-type inequalities involving the above functional.

Theorem 7. *Let $h, g : [\alpha, \beta] \to \mathbb{R}$ be two absolutely continuous functions with $(\cdot - \alpha)(\beta - \cdot)(h')^2, (\cdot - \alpha)(\beta - \cdot)(g')^2 \in L([\alpha, \beta])$. Then*

$$|T(h,g)| \leq \frac{1}{\sqrt{2}} |T(h,h)|^{\frac{1}{2}} \frac{1}{\sqrt{\beta - \alpha}} \left(\int_\alpha^\beta (s-\alpha)(\beta - s)[g'(s)]^2 ds \right)^{\frac{1}{2}}. \tag{34}$$

The constant $\frac{1}{\sqrt{2}}$ is the best possible in (34).

Theorem 8. *Assume that $g : [\alpha, \beta] \to \mathbb{R}$ is monotonic non-decreasing and $h : [\alpha, \beta] \to \mathbb{R}$ is absolutely continuous with $h' \in L^\infty([\alpha, \beta])$. Then*

$$|T(h,g)| \leq \frac{1}{2(\beta - \alpha)} \|h'\|_\infty \int_\alpha^\beta (s-\alpha)(\beta - s) dg(s). \tag{35}$$

The constant $\frac{1}{2}$ is the best possible in (35).

To simplify notation, for $\omega \in \{1,2,3,4\}$, we introduce functions $P_{1,\omega}, P_{2,\omega} : [\alpha, \beta] \to \mathbb{R}$. We assume that all the terms appearing in $P_{1,\omega}$ and $P_{2,\omega}$ satisfy the assumptions of Theorem 4.

$$P_{1,w}(s) = \int_{\alpha}^{\beta} \left[\int\limits_{\Omega_2} G_\omega(f(y), t) v(y) d\mu_2(y) - \int\limits_{\Omega_1} G_\omega(A_k f(x), t) u(x) d\mu_1(x) \right] \cdot \tilde{T}_{n-2}(t, s) dt, \tag{36}$$

$$P_{2,w}(s) = \int_{\alpha}^{\beta} \left[\int\limits_{\Omega_2} G_\omega(\phi(f(y), t)) v(y) d\mu_2(y) - \int\limits_{\Omega_1} G_\omega(A_k f(x), t) u(x) d\mu_1(x) \right] \cdot T_{n-2}(t, s) dt, \tag{37}$$

Theorem 9. *Let $n \in \mathbb{N}$, $n \geq 4$, $P_{1,\omega}$ and $P_{2,\omega}$ be as in (36) and (37) and $\phi : [\alpha, \beta] \to \mathbb{R}$ be such that $\phi^{(n)}$ is absolutely continuous with $(\cdot - \alpha)(\beta - \cdot)(\phi^{(n+1)})^2 \in L([\alpha, \beta])$. Then*

(i) If $(\cdot - \alpha)(\beta - \cdot)(P'_{1,\omega})^2 \in L([\alpha, \beta])$, the remainder

$$\begin{aligned} \kappa^1(\phi; \alpha, \beta) = & \int\limits_{\Omega_2} \phi(f(y)) v(y) d\mu_2(y) - \int\limits_{\Omega_1} \phi(A_k f(x)) u(x) d\mu_1(x) \\ & - \int_{\alpha}^{\beta} \left[\int\limits_{\Omega_2} G_\omega(f(y), t) v(y) d\mu_2(y) - \int\limits_{\Omega_1} G_\omega(A_k f(x), t) u(x) d\mu_1(x) \right] \\ & \times \sum_{k=1}^{n-1} \frac{k}{(k-1)!} \left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1} - \phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta - \alpha} \right) dt \\ & - \frac{\phi^{(n-1)}(\beta) - \phi^{(n-1)}(\alpha)}{(n-3)!(\beta - \alpha)} \int_{\alpha}^{\beta} P_{1,\omega}(s) ds \end{aligned} \tag{38}$$

is bounded by

$$\left| \kappa^1(\phi; \alpha, \beta) \right| \leq \frac{\sqrt{\beta - \alpha}}{\sqrt{2}(n-3)!} |T(P_{1,\omega}, P_{1,\omega})|^{\frac{1}{2}} \left(\int_{\alpha}^{\beta} (s-\alpha)(\beta - s)[\phi^{(n+1)}(s)]^2 ds \right)^{\frac{1}{2}}. \tag{39}$$

(ii) *If* $(\cdot - \alpha)(\beta - \cdot)(P'_{2,\omega})^2 \in L([\alpha, \beta])$*, the remainder*

$$\begin{aligned}\kappa^2(\phi;\alpha,\beta) = &\int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) \\ &- \frac{1}{2}\frac{\phi'(\beta)-\phi'(\alpha)}{\beta-\alpha}\left[\int_{\Omega_2} v(y)f^2(y)d\mu_2(y) - \int_{\Omega_1} u(x)A_k^2 f(x)d\mu_1(x)\right] \\ &- \int_\alpha^\beta \left[\int_{\Omega_2} G_\omega(f(y),t)v(y)d\mu_2(y) - \int_{\Omega_1} G_\omega(A_k f(x),t)u(x)d\mu_1(x)\right] \times \\ &\sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1} - \phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta-\alpha}\right)dt \\ &- \frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{(n-3)!(\beta-\alpha)}\int_\alpha^\beta P_{2,\omega}(s)ds,\end{aligned} \tag{40}$$

is bounded by

$$\left|\kappa^2(\phi;\alpha,\beta)\right| \le \frac{\sqrt{\beta-\alpha}}{\sqrt{2}(n-3)!}|T(P_{2,\omega},P_{2,\omega})|^{\frac{1}{2}}\left(\int_\alpha^\beta (s-\alpha)(\beta-s)[\phi^{(n+1)}(s)]^2 ds\right)^{\frac{1}{2}}. \tag{41}$$

Proof.

(i) From (21) and (38), we conclude

$$\kappa^1(\phi;\alpha,\beta) = \frac{1}{(n-3)!}\int_\alpha^\beta P_{1,\omega}(s)\phi^{(n)}(s)ds - \frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{(n-3)!(\beta-\alpha)}\int_\alpha^\beta P_{1,\omega}(s)ds. \tag{42}$$

The assumptions of Theorem 7 are satisfied for $h = P_{1,\omega}$ and $g = \phi^{(n)}$, so

$$\begin{aligned}&\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta P_{1,\omega}(s)\phi^{(n)}(s)ds - \frac{1}{\beta-\alpha}\int_\alpha^\beta P_{1,\omega}(s)ds \cdot \frac{1}{\beta-\alpha}\int_\alpha^\beta \phi^{(n)}(s)ds\right| \\ &\quad \le \frac{1}{\sqrt{2}}|T(P_{1,\omega},P_{1,\omega})|^{\frac{1}{2}}\frac{1}{\sqrt{\beta-\alpha}}\left(\int_\alpha^\beta (s-\alpha)(\beta-s)[\phi^{(n+1)}(s)]^2 ds\right)^{\frac{1}{2}}.\end{aligned} \tag{43}$$

Therefore, from (42) and (43), we obtain (39).

(ii) Similarly as in part (i), we obtain (41).

□

Theorem 10. *Let* $n \in \mathbb{N}$, $n \ge 4$, $P_{1,\omega}$ *and* $P_{2,\omega}$ *be as in (36) and (37) and* $\phi : [\alpha,\beta] \to \mathbb{R}$ *be such that* $\phi^{(n)}$ *is monotonic non-decreasing. Then*

(i) *If* $P_{1,\omega}$ *is absolutely continuous with* $P'_{1,\omega} \in L^\infty([\alpha,\beta])$*, then the remainder* $\kappa^1(\phi;\alpha,\beta)$ *given by (38) is bounded by*

$$\begin{aligned}&\left|\kappa^1(\phi;\alpha,\beta)\right| \le \\ &\frac{\|P'_{1,\omega}\|_\infty}{(n-3)!}\left[\frac{(\beta-\alpha)\left(\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)\right)}{2} - \left\{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)\right\}\right].\end{aligned} \tag{44}$$

(ii) *If $P_{2,\omega}$ is absolutely continuous with $P'_{2,\omega} \in L^{\infty}([\alpha, \beta])$, then the remainder $\kappa^2(\phi; \alpha, \beta)$ given by (40) is bounded by*

$$\left|\kappa^2(\phi;\alpha,\beta)\right| \\ \leq \frac{\|P'_{2,\omega}\|_{\infty}}{(n-3)!}\left[\frac{(\beta-\alpha)\left(\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)\right)}{2}-\left\{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)\right\}\right]. \quad (45)$$

Proof.

(i) Assumptions of Theorem 8 are satisfied for $h = P_{1,\omega}$ and $g = \phi^{(n)}$, so, taking into account (42), we have

$$\left|\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}P_{1,\omega}(s)\phi^{(n)}(s)ds-\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}P_{1,\omega}(s)ds\cdot\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\phi^{(n)}(s)ds\right| \\ \leq \frac{1}{2(\beta-\alpha)}\|P'_{1,\omega}\|_{\infty}\int_{\alpha}^{\beta}(s-\alpha)(\beta-s)\phi^{(n+1)}(s)ds. \quad (46)$$

Simple calculation yields

$$\int_{\alpha}^{\beta}(s-\alpha)(\beta-s)\phi^{(n+1)}(s)ds=\int_{\alpha}^{\beta}[2s-(\alpha+\beta)]\phi^{(n)}(s)ds \\ =(\beta-\alpha)\left[\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)\right]-2\left[\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)\right].$$

Finally, inserting the last expression in (46) and taking into account (42), we obtain (44).

(i) Similarly as in part (i), we obtain (45).

□

The last theorem gives an Ostrowki-type bound for the generalized Hardy's inequality. The symbol $\|\cdot\|_p$ denotes the standard $L^p([\alpha,\beta])$ norm of a function, i.e.,

$$\| g \|_p=\left(\int_{\alpha}^{\beta}|g(s)|^p ds\right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$, while $\|g\|_{\infty}$ is the essential supremum of g.

Theorem 11. *Let $n \in \mathbb{N}$, $n \geq 4$, $P_{1,\omega}$ and $P_{2,\omega}$ be as in (36) and (37), $1 \leq p,q \leq \infty$, $1/p+1/q=1$ and $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\|\phi^{(n)}\|_p < \infty$. Then*

(i)

$$\left|\int\limits_{\Omega_2}\phi(f(y))v(y)d\mu_2(y)-\int\limits_{\Omega_1}\phi(A_kf(x))u(x)d\mu_1(x)\right. \\ -\int_{\alpha}^{\beta}\left[\int\limits_{\Omega_2}G_{\omega}(f(y),t)v(y)d\mu_2(y)-\int\limits_{\Omega_1}G_{\omega}(A_kf(x),t)u(x)d\mu_1(x)\right]\times \\ \left.\sum_{k=1}^{n-1}\frac{k}{(k-1)!}\left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1}-\phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta-\alpha}\right)dt\right| \\ \leq \frac{1}{(n-3)!}\left\|\phi^{(n)}\right\|_p\|P_{1,\omega}\|_q.$$

The constant $\|P_{1,\omega}\|_q/(n-3)!$ is sharp when $1 < p \leq \infty$ and the best possible when $p = 1$.

(ii)

$$
\begin{aligned}
&\left| \int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x) \right. \\
&- \frac{1}{2}\frac{\phi'(\beta)-\phi'(\alpha)}{\beta-\alpha}\left[\int_{\Omega_2} v(y)f^2(y)d\mu_2(y) - \int_{\Omega_1} u(x)A_k^2 f(x)d\mu_1(x)\right] \\
&- \left[\int_{\Omega_2} G_\omega(f(y),t)v(y)d\mu_2(y) - \int_{\Omega_1} G_\omega(A_k f(x),t)u(x)d\mu_1(x)\right] \times \\
&\left. \sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\left(\frac{\phi^{(k)}(\alpha)(t-\alpha)^{k-1} - \phi^{(k)}(\beta)(t-\beta)^{k-1}}{\beta-\alpha}\right)dt \right| \\
&\leq \frac{1}{(n-3)!}\left\|\phi^{(n)}\right\|_p \|P_{2,\omega}\|_q
\end{aligned}
$$

The constant $\|P_{2,\omega}\|_q/(n-3)!$ *is sharp when* $1 < p \leq \infty$ *and the best possible when* $p = 1$.

Proof. The proof is similar to the proof of Theorem 12 in [21]. □

4. Discussion

We have presented new results regarding Hardy's inequality in a general setting. The main results involve Hardy-type inequalities and four new Green functions. We were motivated by the results given in papers [15–17]. These papers contain results involving the Hardy inequality and Taylor's formula and also results involving the Hardy inequality and Abel–Gontscharoff's interpolating polynomial. We have also derived related Grüss and Ostrowski-type inequalities. In our next papers, we plan to present new results involving Hardy's inequality and Hermite's and Lidstone's interpolating polynomials. The results presented here are of theoretical nature, and any suggestions for possible applications and further research are welcome.

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