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Global forcing number for maximal matchings

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ABSTRACT

Let $\mathcal{M}(G)$ denote the set of all maximal matchings in a simple graph G, and $f : \mathcal{M}(G) \to \{0, 1\}^{|E(G)|}$ be the characteristic function of maximal matchings of G. Any set $S \subseteq E(G)$ such that $f|_S$ is an injection is called a global forcing set for maximal matchings in G, and the cardinality of smallest such S is called the global forcing number for maximal matchings of G. In this paper we establish sharp lower and upper bounds on this quantity and prove explicit formulas for certain classes of graphs. At the end, we also state some open problems and discuss some further developments.

1. Introduction and motivation

The concept of **forcing set** is one of many graph-theoretical concepts whose origins can be traced back to the study of resonance structures in mathematical chemistry where it was introduced under the name of the innate degree of freedom [10,12]. Later it attracted significant attention also in purely graph-theoretical literature [1,2,14,15,23]. The forcing sets were first defined locally, with reference to particular Kekulé structures (or perfect matchings in mathematical literature), and global results were obtained by considering extremal values over the set of all relevant structures. Then the focus shifted to the study of forcing sets that were defined globally in a graph, motivated by the need to efficiently code and manipulate perfect matchings in large-scale computations [19,20]. It turned out that many results could be successfully transferred from the local to the global context. In particular, explicit formulas for the global forcing number for some benzenoid graphs, rectangular and triangular grids and complete graphs were obtained by some of the present authors [5,16,18,21].

Instrumental in obtaining those results were the elements of well-developed structural theory available for perfect matchings. No such theory, however, exists for much less researched but still very useful and interesting class of large matchings, known as maximal matchings. Hence, we were unable to simply transfer the above results when a need for analogous concepts arose in course of our work on maximal matchings. The aim of this paper is to fill the gap by extending the concepts of global forcing set and global forcing number also to maximal matchings and to obtain results analogous to those mentioned for the perfect matching case.

The paper is organized as follows. In the next section we define the terms relevant for our subject and present some preliminary results. Section 3 contains some lower bounds on the global forcing number and also a monotonicity results used later. Sections 4 and 5 present results on trees and complete graphs, respectively, while in Section 6 we present bounds

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for graphs of a given cyclomatic number. Finally, in the concluding section we comment on some open problems and indicate some possible directions for future research.

2. Definitions and preliminary results

All graphs in this paper are tacitly supposed to be simple and connected unless explicitly stated otherwise. Let G be a graph with set of vertices $V(G)$ and set of edges $E(G)$. We will denote by $n = |V(G)|$ the number of vertices and by $m = |E(G)|$ the number of edges in G. As usual, the path, the star, and the complete graph on n vertices are denoted by P_n, S_n and K_n, respectively.

Let G be a graph and H be any subgraph of G. We denote by $G - H$ the graph obtained by deleting from G all vertices of H and all edges incident with them. If S is a set of edges of G, then $G - S$ denotes the graph obtained from G by removing edges from S without removing their end-vertices. We reserve notation $S \setminus M$ for the set difference of two sets of edges.

A connected graph G is **acyclic** or a **tree** if G does not contain cycles. If G contains exactly one cycle, we say G is an **unicyclic** graph. Finally, for any graph G we define its **cyclomatic number** $c(G)$ by $c(G) = |E(G)| - |V(G)| + 1$. That is the smallest number of edges one must remove from a graph to obtain a tree. If G is a tree, then $c(G) = 0$, and if G is an unicyclic graph, then $c(G) = 1$. A vertex u in a graph G is a **leaf** if u has exactly one neighbor. The only neighbor of a leaf in G is called a **petal**.

A **matching** in a graph G is any set of edges $M \subseteq E(G)$ such that every vertex in G is incident with at most one edge from M. The number of edges in M is called its **size**. Matchings of small size are quite uninteresting, since they are easy to construct and enumerate. On the other hand, "large" matchings serve as models for many problems in which we have entities capable of interactions over a given connection pattern. Whenever one neighbor can monopolize all interaction capability of an entity, rendering it unavailable for its other neighbors, matchings naturally appear, and existence of "large" matchings is usually desirable as it signals good efficiency of the underlying process. Hence, we are interested in study of large matchings.

A matching M is **maximum** if there is no matching in G of a greater size. The cardinality of any maximum matching in G is called the **matching number** of G and denoted by $\nu(G)$. Since each edge of a matching saturates two vertices of G, no matching in G can have size greater than $\lfloor n/2 \rfloor$. We say that a matching M is **perfect** if every vertex from G is incident with exactly one edge from M. Obviously, only graphs on an even number of vertices can have perfect matchings. If a graph G on an odd number of vertices has $\nu(G) = \lfloor n/2 \rfloor$, we say that G has an **almost perfect matching**.

Another way of measuring how large is a given matching is based on (im)possibility of its extension to a larger matching. We say that matching M is **maximal** if there is no matching M' in G such that $M \subset M'$. Note that every maximum matching in G is also maximal, but the opposite is, in general, not true. Maximal matchings usually come in different sizes. The smallest size of a maximal matching in G is called the **saturation number** of G and denoted by $s(G)$. The largest size is, of course, the matching number $\nu(G)$. If all maximal matchings in G are of the same size (and hence maximum), graph G is **equimatchable**.

There is a marked asymmetry in the way maximum and maximal matchings are studied and represented in the literature. While the maximum (and in particular perfect) matchings are well researched and understood (see, for example, monographs [13] and [4]), results on their maximal counterparts are much less abundant. We mention here some papers dealing with maximal matchings in trees [11,22], with equimatchable graphs [8,9], and two recent papers coauthored by one of the present authors about structural and enumerative aspects of maximal matchings in linear polymers [6,7]. One of possible reasons for the scarcity of results might be that, at the moment, there is no structural theory for maximal matchings analogous to the one available for maximum matchings.

Any non-maximal matching can be extended to a maximal matching. In particular, for any edge $e \in E(G)$ there is a maximal matching M containing e. This stands in sharp contrast with the situation for perfect matchings, where such property (1-extendability) imposes strong structural conditions on G. The idea of finding a subset of a perfect matching which is in a unique way extendable to the whole matching gave rise to the concept of forcing set.

For a given perfect matching M in G, *its* forcing set is defined as any subset of M that is not contained in any other perfect matching of G. The forcing number of a perfect matching M was defined as the size of any smallest forcing set of M. Note that forcing sets and numbers are defined for each perfect matching of G. The idea was generalized to global setting in two different ways. One was to study extremal forcing sets and forcing numbers over all perfect matchings; the other was to look for subsets of edges of G, not necessarily matchings, such that no two perfect matchings coincide on them. The later approach gave rise to the concept of global forcing sets and numbers for perfect matchings. Now we extend the idea also to maximal matchings.

A **global forcing set for maximal matchings** of a graph G is any set $S \subseteq E(G)$ such that $M_1\big|_S \neq M_2\big|_S$ for any two maximal matchings M_1 and M_2. Any global forcing set for maximal matchings in G of the smallest cardinality is called a minimum global forcing set and its cardinality, denoted by $\varphi_{gm}(G)$, is called the **global forcing number for maximal matchings** in G. Throughout the rest of the paper we will say only global forcing set (or number) of graph G tacitly assuming it is a global forcing set (or number) for maximal matchings in G unless explicitly stated otherwise.

Global forcing sets in a given graph G have an obvious monotonicity property.

Proposition 1. *If $S \subset E(G)$ is a global forcing set, then each $S' \supset S$ is also a global forcing set. If $S \subset E(G)$ is not a global forcing set, then no $S' \subset S$ can be a global forcing set.*

Let $S = \{e_1, \ldots, e_k\}$ be a global forcing set for maximal matchings in G. To each maximal matching M we associate a binary string of length k by setting its ith bit to 1 if $e_i \in M$ and to 0 otherwise. (Those binary strings are, in fact, restrictions of the characteristic functions of maximal matchings on S.) Since different matchings give rise to different strings, the number of bits must be sufficiently large to provide different binary codes for all maximal matchings. This gives a lower bound on the global forcing number for maximal matchings.

Proposition 2. *Let $\Psi(G)$ be the total number of maximal matchings in G. Then $\varphi_{gm}(G) \geq \lceil \log_2 \Psi(G) \rceil$.*

A completely analogous result is valid for global forcing number for perfect matchings (Proposition 1 in [18]). The logarithmic nature of the lower bound is reinforced by noticing that it becomes sharp for graphs with unique maximal matchings, since in such graphs any subset of edges, including the empty one, is a global forcing set. In the case of perfect matchings, analogous observation provided an important characterization of global forcing sets (Proposition 5 of [5]). Here, however, it is far less useful. One possible reason is that the class of graphs with unique maximal matchings is quite restricted.

Proposition 3. *Let G be a graph on n vertices with only one maximal matching. Then G is a union of a matching and an empty graph. More precisely, $G = pK_2 \cup qK_1$ for some integer p and q such that $2p + q = n$.*

We can now use this result to find global forcing sets.

Proposition 4. *Let $S \subseteq E(G)$ be a set of edges such that the graph induced by $E(G) \setminus S$ has only one maximal matching. Then S is a global forcing set for maximal matchings.*

The converse, however, is not true. It suffices to consider the cycle on 4 vertices. It has 2 maximal matchings, both of them perfect, and they differ on every edge. Hence, each edge is a global forcing set, and the graph induced by the remaining three edges has two maximal matchings, one of size one and one of size two.

An immediate consequence of the above result is that the complement of any matching is a global forcing set.

Corollary 5. *Let M be any matching in G. Then $E(G) \setminus M$ is a global forcing set for maximal matchings in G.*

We will need another result on global forcing sets for perfect matchings in a graph. Again, let G be a connected graph with a perfect matching. A subgraph H of G is **nice** if $G - H$ contains a perfect matching. We refer the reader to [3] for the proof of the following result.

Theorem 6. *Let G be a graph with a perfect matching. Then $S \subseteq E(G)$ is a global forcing set for perfect matchings in G if and only if S intersects each nice cycle of G.*

We close the section by introducing some special graphs which will be used throughout the paper. A **cycle-star graph** $CS_{k,n-k}$ is a graph on n vertices consisting of one cycle on k vertices and $n-k$ leaves all adjacent to the same vertex of the cycle. A **cycle-path graph** $CP_{k,n-k}$ is a graph on n vertices consisting of one cycle on k vertices and one path on $n-k$ vertices which are vertex disjoint and one end of the path is connected by an edge to exactly one vertex on the cycle. Finally, we define a **multicycle-path graph** $CP_{c,k,n-ck}$ as a graph on n vertices consisting of c vertex disjoint cycles on k vertices and one path on $n - ck$ vertices vertex disjoint with those c cycles, such that one end of the path is connected to exactly one vertex of each cycle by an edge. It follows from the definition that $CP_{1,k,n-k} = CP_{k,n-k}$.

3. Upper bound and monotonicity

In this section we use the above observations to establish an upper bound on the global forcing number and also prove that the global forcing number is monotonously increasing with respect to addition of edges.

Theorem 7. *Let G be a simple graph on n vertices and m edges. Then $\varphi_{gm}(G) \leq m - \nu(G)$.*

The result follows directly from Corollary 5 by noticing that taking a larger matching M results in a smaller complement. Since we are interesting in small global forcing sets, it pays off to choose M as large as possible, hence of size $\nu(G)$.

It is easy to see that the upper bound of Theorem 7 is not sharp for some graphs. An example is provided by C_4. Another example is K_4 in which any set of two edges incident with same vertex is a global forcing set. It is easy to see that all three maximal (and also perfect) matchings differ on this set. Both examples are randomly matchable i.e. every maximal matching is also perfect; K_{2n} and $K_{n,n}$ are the only randomly matchable graphs for any n (see [17]). However, $K_4 - e$ is a further example, showing that the bound is not tight on a wider class of graphs. It is an interesting problem to determine for which classes of graphs is this upper bound tight and also to find out how far from the exact value it can be. We will show that the upper bound is attained for all trees and all complete bipartite graphs with unequal classes of bipartition.

Next we show that the global forcing number does not decrease with an edge addition.

Theorem 8. *Let G be a graph on n vertices and let u, v be two vertices in G such that $uv \notin E(G)$. Let G_1 be the graph obtained from G by adding the edge $e = uv$ to G. Then $\varphi_{gm}(G) \leq \varphi_{gm}(G_1)$.*

Proof. Suppose the contrary, i.e. $\varphi_{gm}(G_1) < \varphi_{gm}(G)$. Let S_1 be a global forcing set in G_1. Therefore, $|S_1| \le \varphi_{gm}(G) - 1$. We distinguish two cases.

CASE 1. Suppose S_1 does not contain the edge $e = uv$. Therefore, $S_1 \subseteq E(G)$. Since $|S_1| \le \varphi_{gm}(G) - 1$, we know that S_1 cannot be a global forcing set in G. That implies there are two different maximal matchings M and M' in G such that $M\big|_{S_1} = M'\big|_{S_1}$. Note that maximal matchings M and M' can be extended to maximal matchings M_1 and M_1' in G_1. Therefore, we obtained two different maximal matchings M_1 and M_1' in G_1, such that $M_1\big|_{S_1} = M_1'\big|_{S_1}$ which is contradiction with S_1 being a global forcing set on G_1.

CASE 2. Suppose S_1 contains the edge $e = uv$. Let $S = S_1 \setminus \{e\}$. Therefore, $S \subseteq E(G)$ and $|S| \le \varphi_{gm}(G) - 2$. Since $|S| \le \varphi_{gm}(G) - 2$, we know that S cannot be a global forcing set in G. That implies there are two different maximal matchings M and M' in G such that $M\big|_S = M'\big|_S$. If M and M' were the only two different maximal matchings in G coinciding on S, then we could pick an edge $e' \in E(G) \setminus S$ on which M and M' differ, and define $S' = S \cup \{e\}$. Obviously, S' would be a global forcing set on G with $|S'| = |S| + 1 \le \varphi_{gm}(G) - 1$ which is contradiction with $\varphi_{gm}(G)$ being the global forcing number of G. Therefore, there must exist a maximal matching M'' in G different from both M and M' such that $M\big|_S = M'\big|_S = M''\big|_S$. Note that all three maximal matchings M, M' and M'' in G can be extended to maximal matchings M_1, M_1' and M_1'' in G_1, respectively. The extension of a maximal matching M in G to maximal matching M_1 in G_1 can be done in only two ways: $M_1 = M \cup \{e\}$ or $M_1 = M$ (which way M will be extended depends on whether M saturates at least one of vertices u and v or not). Since there are only two ways to extend maximal matchings from G to G_1, at least two of maximal matchings M, M' and M'' are extended in the same way. Without loss of generality we may assume those two matchings are M and M'. Therefore, we have obtained two different maximal matchings M_1 and M_1' in G_1 such that $M_1\big|_{S_1} = M_1'\big|_{S_1}$ which is contradiction with S_1 being a global forcing set on G_1. ■

4. The global forcing number for trees

In this section we prove that all trees satisfy the upper bound of Theorem 7 with equality.

Lemma 9. *Let T_n be a tree on n vertices. Then $\varphi_{gm}(T_n) \ge n - 1 - \nu(T_n)$.*

Proof. Let S be a global forcing set for maximal matchings in T_n. We consider the graph $T_n - S$. Let $P = v_0v_1 \ldots v_k$ be the longest path in $T_n - S$. Since P is the longest, it cannot be extended, and it follows that all edges incident with v_0 and v_k in $T_n - P$ are included in S. There are two cases to consider.

CASE 1. Suppose $k = 1$. Then $E(T_n) - S$ must be a matching in T_n. Therefore, $|E(T_n)| - |S| \le \nu(T_n)$, which implies $|S| \ge |E(T_n)| - \nu(T_n)$.

CASE 2. Suppose $k > 1$.

CASE 2a. Suppose neither v_0 nor v_k is a petal in T_n. Let us denote edges in path P by $e_i = v_{i-1}v_i$ for $i = 1, \ldots, k$. Now, let $M_1' = \{e_{2i-1} : i = 1, \ldots, \lceil \frac{k}{2} \rceil\}$ and $M_2' = \{e_{2i} : i = 1, \ldots, \lfloor \frac{k}{2} \rfloor\}$. Note that M_1' and M_2' are two different maximal matchings in P, both saturating all interior vertices of P. Now, let us consider the graph $T_n - P$. If a connected component of $T_n - P$ is an isolated vertex u, then u is a leaf in T_n adjacent to an interior vertex of P, so M_1' and M_2' cannot be extended to a maximal matching in T_n which saturates those leaves. If a connected component of $T_n - P$ is not an isolated vertex, then let u be the only vertex in that connected component which in T_n is a neighbor to a vertex in P. We call u the **root** of that connected component. Since every edge in a connected graph can be extended to a maximal matching, there exists maximal matching M' in $T_n - P$ which saturates all roots of connected components. Therefore, $M_1 = M_1' \cup M'$ and $M_2 = M_2' \cup M'$ are two different maximal matchings in T_n coinciding on S, which is a contradiction with S being a global forcing set for maximal matchings.

CASE 2b. Suppose v_0 or v_k is a petal in T_n. Without loss of generality we may assume that v_0 is a petal adjacent to a leaf w in T_n. Let us consider set $S' = (S \cup \{v_0v_1\}) \setminus \{v_0w\}$. Note that $|S| = |S'|$ and S' is also a global forcing set for maximal matchings (since all edges in T_n incident to v_0 are included in S except v_0v_1). Also, note that the way we changed S for S' means that at least one longest path in $T_n - S$ no longer exists in $T_n - S'$ (i.e. it is shortened). Therefore, applying this procedure enough times we will reduce this case to CASE 1 or CASE 2a.

Note that in each case we have proved that for any global forcing set S there is a global forcing set S' such that $|S| = |S'| \ge |E(T_n)| - \nu(T_n)$. Therefore, we have

$$\varphi_{gm}(T_n) = \min\{|S| : S \text{ is a global forcing set of } T_n\} \ge |E(T_n)| - \nu(T_n),$$

which proves the lemma. ■

The main result of this section now follows immediately.

Theorem 10. *Let T_n be a tree on n vertices. Then $\varphi_{gm}(T_n) = n - 1 - \nu(T_n)$.*

Several corollaries now follow by plugging in expressions for matching number of some classes of trees.

Corollary 11. $\varphi_{gm}(S_n) = n - 2$.

Corollary 12. $\varphi_{gm}(P_n) = \lceil \frac{n}{2} \rceil - 1$.

Theorem 13. *Let T_n be a tree on n vertices. Then*

$$\left\lceil \frac{n}{2} \right\rceil - 1 \leq \varphi_{gm}(T_n) \leq n - 2.$$

The lower bound is achieved for any tree on n vertices with matching number equal to $\lfloor \frac{n}{2} \rfloor$, while the upper bound is attained if and only if T_n is the star S_n.

We conclude the section by noticing that the upper bound for trees can be further tightened in a way stated in the following proposition.

Proposition 14. *Let T_n be a tree on $n > 2$ vertices and let k be the number of petals in T_n. Then $\varphi_{gm}(T_n) \leq n - 1 - k$.*

Proof. It is sufficient to find a global forcing set S in T_n such that $|S| = n - 1 - k$. Let $K = \{u_1, \dots, u_k\}$ be the set of all petals in a tree T_n. Let $L = \{v_1, \dots, v_k\}$ be the set of leaves from T_n such that leaf v_i is a neighbor of the petal u_i. Finally, let $M = \{u_i v_i : i = 1, \dots, k\}$. Obviously, M is a matching in T_n such that $|M| = k$. Corollary 5 implies $S = E(T_n) \setminus M$ is a global forcing set in T_n. Since $|S| = |E(T_n)| - |M| = n - 1 - k$, we obtain $\varphi_{gm}(T_n) \leq n - 1 - k$. ■

5. The global forcing number for complete graphs

Our monotonicity result implies that the largest possible value of the global forcing number on all graphs on a given number of vertices must be attained on the complete graph. We have seen examples indicating that the upper bound of Theorem 7 is not sharp for randomly matchable graphs, hence also for the complete graph on $2n$ vertices. On the other hand, the values of 1 and 8, respectively, for K_3 and K_5, indicate that it might be attained for odd n. Our next result confirms this by providing exact values for the global forcing number of complete graphs.

Lemma 15. *Let K_n be a complete graph on an odd number of vertices n. Let M be a maximal matching in K_n and let $S = E(K_n) \setminus M$. Then S is a minimum global forcing set in K_n.*

Proof. Corollary 5 implies that S is a global forcing set. Therefore, $\varphi_{gm}(K_n) \leq \frac{n(n-1)}{2} - \lfloor \frac{n}{2} \rfloor$. Suppose now that there is a global forcing set S in K_n such that $|S| < \frac{n(n-1)}{2} - \lfloor \frac{n}{2} \rfloor$. Then, there is at least one vertex u in K_n which is incident to at least two edges not belonging to S. Let us denote by v and w two neighbors of u such that $uv \notin S$ and $uw \notin S$. Let P be the path vuw. Note that vertices on path P can be maximally matched in two different ways and that $K_n - P$ has even number of vertices which can be perfectly matched in $K_n - P$. Therefore, there are two different maximal matchings in K_n which differ only on edges uv and uw which do not belong to S. We conclude that S cannot be a global forcing set. ■

Lemma 16. *Let K_n be a complete graph on an even number of vertices n. Let G be a subgraph of K_n containing all vertices from K_n, all edges incident to a vertex $u \in V(K_n)$ and a maximal matching in $K_n - \{u\}$. Then $S = E(K_n) \setminus E(G)$ is a minimum global forcing set in K_n.*

Proof. First note that in this case all maximal matchings on K_n are perfect, so a global forcing set for maximal matchings is the same as a global forcing set for perfect matchings. Now we can use the results for perfect matchings in K_n, namely Theorem 6. Since all even cycles in K_n are nice, Theorem 6 implies that S will be a global forcing set in K_n if and only if $K_n - S$ does not contain even cycle. Therefore, S will be a global forcing set of minimum cardinality if and only if $G = K_n - S$ is a graph with maximum possible number of edges that does not contain even cycle. We want to establish $m = |E(G = K_n - S)|$ for such $G = K_n - S$. Since $G = K_n - S$ must not contain even cycles, note that all cycles in $G = K_n - S$ must be edge disjoint. Let c be the cyclomatic number of $G = K_n - S$. The definition of cyclomatic number implies $c \geq m - (n - 1)$. On the other hand, since all cycles in $K_n - S$ must be edge disjoint, it follows that $c \leq \lfloor \frac{m}{3} \rfloor$. Therefore, we obtain

$$m - (n-1) \leq c \leq \left\lfloor \frac{m}{3} \right\rfloor \leq \frac{m}{3} \Rightarrow \frac{2m}{3} \leq n - 1 \Rightarrow m \leq \left\lfloor \frac{3(n-1)}{2} \right\rfloor.$$

The bound is obtained by the graph $G = K_n - S$ consisting of all edges incident to a vertex u and a maximal matching in $K_n - \{u\}$. ■

Theorem 17. *Let G be a graph on n vertices. Then*

$$\left\lceil \frac{n}{2} \right\rceil - 1 \leq \varphi_{gm}(G) \leq \begin{cases} \dfrac{(n-1)^2}{2} & \text{for odd } n, \\ \dfrac{(n-2)^2}{2} & \text{for even } n. \end{cases}$$

Any tree on n vertices with a perfect or an almost perfect matching attains the lower bound, while the complete graph K_n attains the upper bound.

Proof. Let us first prove the claim for the lower bound. Let G be a graph on n vertices and T its spanning tree. Theorem 8 implies $\varphi_{gm}(G) \geq \varphi_{gm}(T)$. Hence no graph on n vertices can have the global forcing number smaller than a tree on the same number of vertices having a perfect or an almost prefect matching. As such trees exist for all n, it follows that the trees with matching number equal to $\lfloor \frac{n}{2} \rfloor$ are among the graphs on n vertices attaining the smallest global forcing number. They do not exhaust the class, though, as can be seen on example of C_4. Other examples of cyclic graphs attaining the lower bound $\lceil \frac{n}{2} \rceil - 1$ will be presented in the next section.

Regarding the upper bound, $\varphi_{gm}(K_n) \geq \varphi_{gm}(G)$ for any G on n vertices follows directly from Theorem 8. Now, Lemma 15 implies that for odd n we have

$$\varphi_{gm}(K_n) = |E(K_n)| - |E(K_n - S)| = \frac{n(n-1)}{2} - \left\lfloor \frac{3(n-1)}{2} \right\rfloor = \frac{n(n-1)}{2} - \frac{3n-4}{2} = \frac{(n-2)^2}{2}.$$

In the case of even n, Lemma 16 implies that we have $\varphi_{gm}(K_n) = \frac{n(n-1)}{2} - \lfloor \frac{n}{2} \rfloor = \frac{(n-1)^2}{2}$. Again, K_n is not the only graph attaining the maximum possible global forcing number. A simple example is provided by $K_4 - e$. ■

In the rest of this section we consider complete bipartite graphs. They are equimatchable (and even randomly matchable for classes of equal size) and there always exists a maximal matching that saturates the smaller class of the bipartition.

Lemma 18. *Let $K_{p,p}$ be a complete bipartite graph on $n = 2p$ vertices. Let $G = K_{p-1,p-1}$ be a subgraph of $K_{p,p}$. Then $S = E(G)$ is a global forcing set of $K_{p,p}$.*

Proof. First, note that every maximal matching in $K_{p,p}$ is a perfect matching. Let us denote the vertices of $K_{p,p}$ by $u_1, \ldots, u_p$ and $v_1, \ldots, v_p$ so that every edge in $K_{p,p}$ is of the form $e = u_i v_j$. Without loss of generality we may assume that a subgraph $G = K_{p-1,p-1}$ of $K_{p,p}$ is induced by set of vertices

$$\{u_1, \ldots, u_{p-1}\} \cup \{v_1, \ldots, v_{p-1}\}.$$

We claim that $S = E(G)$ is a global forcing set for maximal matchings in $K_{p,p}$. Suppose M_1 and M_2 are two different maximal matchings in $K_{p,p}$. Since M_1 and M_2 are of the same size (both are perfect), their difference implies that in $\{u_1, \ldots, u_p\}$ there are at least two vertices, say u_i and u_j, which are differently matched in M_1 and M_2. Since $u_i \neq u_j$, it follows that at least one of those vertices belongs to $V(G = K_{p-1,p-1})$, say u_i. Since u_i is differently matched in M_1 and M_2, let $u_i v_k \in M_1 \setminus M_2$ and $u_i v_l \in M_2 \setminus M_1$. Without loss of generality we may assume $k < l$. Therefore, M_1 and M_2 differ on the edge $u_i v_k \in S$, so we have proved that $M_1\big|_S \neq M_2\big|_S$, which means that S is the global forcing set for maximal matching. ■

Lemma 19. *Let $K_{p,q}$ be a complete bipartite graph on $n = p + q$ vertices, where $p < q$. Let $G = K_{p,q-1}$ be a subgraph of $K_{p,q}$. Then $S = E(G)$ is a global forcing set of $K_{p,q}$.*

Proof. Let us denote the vertices of $K_{p,q}$ by $u_1, \ldots, u_p$ and $v_1, \ldots, v_q$ so that every edge in $K_{p,q}$ is of the form $e = u_i v_j$. Note that every maximal matching in $K_{p,q}$ saturates all vertices from $\{u_1, \ldots, u_p\}$. Without loss of generality we may assume that a subgraph $G = K_{p,q-1}$ of $K_{p,q}$ is induced by set of vertices

$$\{u_1, \ldots, u_p\} \cup \{v_1, \ldots, v_{q-1}\}.$$

We claim that $S = E(G)$ is a global forcing set for maximal matchings in $K_{p,q}$. Suppose M_1 and M_2 are two different maximal matchings in $K_{p,q}$. Since M_1 and M_2 both saturate all vertices from $\{u_1, \ldots, u_p\}$, their difference implies that in $\{u_1, \ldots, u_p\}$ there is at least one vertex, say u_i, which is differently matched in M_1 and M_2. Suppose $u_i v_k \in M_1 \setminus M_2$ and $u_i v_l \in M_2 \setminus M_1$. Without loss of generality we may assume $k < l$. Therefore, M_1 and M_2 differ on edge $u_i v_k \in S$, so we have proved that $M_1\big|_S \neq M_2\big|_S$, which means that S is the global forcing set for maximal matching. ■

Theorem 20. *Let $K_{p,q}$ be a complete bipartite graph on $n = p + q$ vertices, where $p \leq q$. Then*

$$\varphi_{gm}(K_{p,q}) = \begin{cases} (p-1)^2 & \text{if } p = q \\ p(q-1) & \text{if } p < q \end{cases}.$$

Proof. Let us denote the vertices of $K_{p,q}$ by $u_1, \ldots, u_p$ and $v_1, \ldots, v_q$ so that every edge in $K_{p,q}$ is of the form $e = u_i v_j$. We now distinguish two cases.

CASE 1. Suppose $p = q$. Lemma 18 implies $\varphi_{gm}(K_{p,p}) \leq (p-1)^2$. Now, suppose that $\varphi_{gm}(K_{p,p}) < (p-1)^2$, i.e., there is a global forcing set S in $K_{p,p}$ such that $|S| < (p-1)^2$. Let us consider the graph $G = K_{p,p} - S$. Note that

$$|E(G)| = \left|E(K_{p,p})\right| - |S| > p^2 - (p-1)^2 = 2p - 1 = n - 1,$$

which implies $|E(G)| \geq n$. Therefore, G must contain a cycle C, and since G is a subgraph of $K_{p,p}$, cycle C must be even with the same number of vertices in $\{u_1, \ldots, u_p\}$ as in $\{v_1, \ldots, v_p\}$. Let M_1' and M_2' be two different perfect matchings in C, let

M be a perfect matching of the vertices in $K_{p,p} - C$. Now, note that $M_1 = M_1' \cup M$ and $M_2 = M_2' \cup M$ are two different perfect matchings in $K_{p,p}$ coinciding on S which contradicts S being a global forcing set for maximal matchings. Therefore, $\varphi_{gm}(K_{p,p}) = (p-1)^2$.

CASE 2. Suppose $p < q$. Lemma 19 implies $\varphi_{gm}(K_{p,q}) \le p(q-1)$. Now, suppose that $\varphi_{gm}(K_{p,q}) < p(q-1)$, i.e., there is a global forcing set S in $K_{p,q}$ such that $|S| < p(q-1)$. Let us consider the graph $G = K_{p,q} - S$. Note that

$$|E(G)| = \left|E(K_{p,q})\right| - |S| > pq - p(q-1) = p,$$

which implies $|E(G)| \ge p+1$. This implies that at least one vertex from $\{u_1, \ldots, u_p\}$ must be of degree 2 in $G = K_{p,q} - S$. Suppose that one vertex is u_i and the two edges incident to it are $e_1 = u_i v_j$ and $e_2 = u_i v_k$. Let M be a maximal matching in the graph $K_{p,q} - \{u_i, v_j, v_k\}$. Note that $M_1 = \{e_1\} \cup M$ and $M_2 = \{e_2\} \cup M$ are two different maximal matchings in $K_{p,q}$ coinciding on S which contradicts S being a global forcing set for maximal matchings. Therefore, $\varphi_{gm}(K_{p,q}) = p(q-1)$. ■

Note that Theorem 20 implies that global forcing sets from Lemmas 18 and 19 are minimum global forcing sets.

6. The global forcing number for graphs with given cyclomatic number

In this section we provide lower and upper bounds on the global forcing number of graphs with a given cyclomatic number and construct some graphs attaining those bounds. We start with unicyclic graphs.

Lemma 21. *Let G be an unicyclic graph on n vertices. Then $\varphi_{gm}(G) \ge n - 1 - \nu(G)$.*

Proof. Let G be an unicyclic graph, and let e be an edge on the only cycle in G. Let $T = G - \{e\}$. Note that T is a tree on n vertices, so Theorem 10 implies $\varphi_{gm}(T) \ge n - 1 - \nu(T)$ which is equivalent to $\varphi_{gm}(T) + \nu(T) \ge n-1$. Theorem 8 implies that $\varphi_{gm}(T) \le \varphi_{gm}(G)$. Since any matching in T is also a matching in G, it follows that $\nu(T) \le \nu(G)$ too. Now we have

$$\varphi_{gm}(G) + \nu(G) \ge \varphi_{gm}(T) + \nu(T) \ge n - 1,$$

which is equivalent to $\varphi_{gm}(G) \ge n - 1 - \nu(G)$, so the lemma is proved. ■

By combining this result with the general upper bound, we obtain that the upper and lower bound for the global forcing number of an unicyclic graph differ only by one, which implies that the global forcing number of an unicyclic graph can obtain only two different integer values which are consecutive.

Corollary 22. *Let G be an unicyclic graph on n vertices. Then $n - 1 - \nu(G) \le \varphi_{gm}(G) \le n - \nu(G)$.*

Next we show that for each $n \ge 5$ there is an unicyclic graph on n vertices attaining the lowest possible value of global forcing number among all graphs on n vertices.

Lemma 23. *Let n be an integer such that $n \ge 5$. Then*

$$\varphi_{gm}(CP_{4,n-4}) = \left\lceil \frac{n}{2} \right\rceil - 1.$$

Proof. From Theorems 13 and 8 it follows that $\varphi_{gm}(CP_{4,n-4}) \ge \left\lceil \frac{n}{2} \right\rceil - 1$. Therefore, it is sufficient to prove that there is a global forcing set S in $CP_{4,n-4}$ such that $|S| = \left\lceil \frac{n}{2} \right\rceil - 1$. Let us denote the vertices in $CP_{4,n-4}$ by $u_1, u_2, \ldots, u_n$ so that $u_{i-1}u_i \in E(CP_{4,n-4})$ for every $i = 2, \ldots, n$ and $u_1u_4 \in E(CP_{4,n-4})$. Let S be a set of edges from $CP_{4,n-4}$ defined by

$$S = \{u_1u_2\} \cup \{u_{2i}u_{2i+1} : 2 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor\}.$$

We have $|S| = \left\lfloor \frac{n-1}{2} \right\rfloor = \left\lceil \frac{n}{2} \right\rceil - 1$. It remains to prove that S is a global forcing set in G. Let us consider two different maximal matchings M_1 and M_2 in $CP_{4,n-4}$. Let us denote $e_1 = u_1u_2 \in S$. There are three possible cases with respect to e_1 belonging to M_1 and M_2.

CASE 1. Suppose $e_1 \in M_1 \Delta M_2$, where $M_1 \Delta M_2$ is a symmetric difference of the maximal matchings M_1 and M_2. Then, obviously $M_1\big|_S \ne M_2\big|_S$.

CASE 2. Suppose $e_1 \in M_1 \cap M_2$. Let G' be the graph obtained from G by deleting the vertices u_1 and u_2. It holds that $G' = P_{n-2}$. Note that $M_1' = M_1\big|_{G'}$ and $M_2' = M_2\big|_{G'}$ are two different maximal matchings on G'. Also, by Corollary 5 we know that $S' = S\big|_{G'}$ is a global forcing set in G'. Therefore, $M_1'\big|_{S'} \ne M_2'\big|_{S'}$, which implies $M_1\big|_S \ne M_2\big|_S$.

CASE 3. Suppose $e_1 \notin M_1 \cup M_2$. Note that in this case the following must also hold: $e_3 = u_3u_4 \notin M_1 \cup M_2$ and $e_2 = u_2u_3 \in M_1 \cap M_2$, otherwise M_1 and M_2 would not be maximal matchings in $CP_{4,n-4}$. Let G' be the graph obtained from G by deleting the vertices u_2 and u_3. Obviously, $G' = P_{n-2}$. Also, just as in the previous case we have that $M_1' = M_1\big|_{G'}$ and $M_2' = M_2\big|_{G'}$ are two different maximal matchings on G', and that $S' = S\big|_{G'}$ is a global forcing set in G'. Therefore, $M_1\big|_S \ne M_2\big|_S$. ■

We notice that we could replace P_{n-4} with any tree with matching number equal to $\lfloor \frac{n-4}{2} \rfloor$ and construct a graph satisfying the lower bound by adding an edge between one of its leaves and a vertex on C_4.

The next result shows that the upper bound $n - \nu(G)$ is also satisfied with equality by some unicyclic graphs on n vertices.

Lemma 24. *Let $n \geq 5$ be an integer. Then $\varphi_{gm}(CS_{3,n-3}) = n - 2$.*

Proof. Let us suppose the contrary, i.e. $\varphi_{gm}(CS_{3,n-3}) \leq n - 3$. We denote by u the only vertex in $CS_{3,n-3}$ of degree greater than two. Let S be a global forcing set on $CS_{3,n-3}$ such that $|S| = \varphi_{gm}(CS_{3,n-3})$. Now, $\varphi_{gm}(CS_{3,n-3}) \leq n-3$ and $\left|E(CS_{3,n-3})\right| = n$ imply that at least three edges from $CS_{3,n-3}$ are not contained in S. If two of those edges are incident to u and a leaf, then S cannot be a global forcing set. If two of those edges are incident to u and a non-leaf, then again S cannot be a global forcing set. The only remaining possibility is that one edge is incident to u and a leaf, the other edge is incident to u and a non-leaf and the third edge is the only edge in $CS_{3,n-3}$ which is not incident to u. But in that case also there are two maximal matchings in $CS_{3,n-3}$ coinciding on S, so S again is not a global forcing set. Therefore, in every case we have a contradiction, so the lemma is proved. ■

Now we can summarize our results for unicyclic graphs.

Theorem 25. *Let G be an unicyclic graph on $n \geq 5$ vertices. Then*

$$\left\lceil \frac{n}{2} \right\rceil - 1 \leq \varphi_{gm}(G) \leq n - 2.$$

The cycle-path graph $CP_{4,n-4}$ and the cycle-star graph $CS_{3,n-3}$ attain the lower and the upper bound, respectively, with equality.

Proof. The lower bound follows from Theorems 13 and 8. Furthermore, Lemma 23 implies that the lower bound is obtained for $CP_{4,n-4}$.

Let us now prove the upper bound. Let G be an unicyclic graph on $n \geq 5$ vertices. Note that G must contain at least two independent edges. To see that, note that if the only cycle in the graph G is of length at least 4 then there are two independent edges on that cycle. If the only cycle in G is a triangle, then $n \geq 5$ implies that there must exist an edge e in G not contained on the triangle. But e is then independent with at least one edge of the triangle. Therefore, there is a matching M in G such that $|M| \geq 2$. Corollary 5 implies that $S = E(G) \setminus M$ is a global forcing set, therefore $\varphi_{gm}(G) \leq |S| = |E(G) \setminus M| \leq n - 2$. Lemma 24 implies that the bound is obtained for $CS_{3,n-3}$. ■

The approach of Lemma 21 can be extended to graphs with larger cyclomatic number. The proof follows along the same lines and we omit the details.

Lemma 26. *Let G be a graph on n vertices with cyclomatic number c. Then $\varphi_{gm}(G) \geq m - c - \nu(G)$.*

Corollary 27. *Let G be a graph on n vertices with cyclomatic number c. Then $m - c - \nu(G) \leq \varphi_{gm}(G) \leq m - \nu(G)$.*

If the cycles are independent (hence, for n sufficiently large with respect to c), we can construct graphs with a given cyclomatic number satisfying the lower bound of Theorem 17 with equality.

Lemma 28. *Let c and n be two integers such that $n \geq 4c + 1$. Then*

$$\varphi_{gm}(CP_{c,4,n-4c}) = \left\lceil \frac{n}{2} \right\rceil - 1.$$

Proof. From Theorems 13 and 8 it follows that $\varphi_{gm}(CP_{c,4,n-4c}) \geq \left\lceil \frac{n}{2} \right\rceil - 1$. Therefore, it is sufficient to prove that there is a global forcing set S in $CP_{c,4,n-4c}$ such that $|S| = \left\lceil \frac{n}{2} \right\rceil - 1$. Recall that $CP_{c,4,n-4c}$ consists of c cycles on 4 vertices and a path on $n - 4c$ vertices. Suppose i–th cycle of $CP_{c,4,n-4c}$ is obtained from the path $u_1^{(i)}u_2^{(i)}u_3^{(i)}u_4^{(i)}$ by connecting its end-vertices by an edge. Also, suppose the path on $n - 4c$ vertices in $CP_{c,4,n-4c}$ is denoted by $v_1v_2 \ldots v_{n-4c}$. Finally, suppose $v_1u_4^{(i)}$ is an edge in $CP_{c,4,n-4c}$ for every $i = 1, \ldots, c$. Now that we have denoted all the vertices and edges in $CP_{c,4,n-4c}$, let us define the set of edges S by

$$S = \{u_1^{(i)}u_2^{(i)}, u_4^{(i)}v_1 : i = 1, \ldots, c\} \cup \{v_{2i}v_{2i+1} : 1 \leq i \leq \left\lfloor \tfrac{n-4c-1}{2} \right\rfloor\}.$$

Note that $|S| = 2c + \left\lfloor \frac{n-4c-1}{2} \right\rfloor = \left\lfloor \frac{n-1}{2} \right\rfloor = \left\lceil \frac{n}{2} \right\rceil - 1$. It remains to prove that S is a global forcing set in $CP_{c,4,n-4c}$. The proof is by induction on c. For $c = 1$, the claim is proved in Lemma 23. Suppose $c \geq 2$. Let us consider two different maximal matchings M_1 and M_2 in $CP_{c,4,n-4c}$. Let us denote $e^{(1)} = u_4^{(1)}v_1 \in S$. There are three possible cases with respect to $e^{(1)}$ belonging to M_1 and M_2.

CASE 1. Suppose $e^{(1)} \in M_1 \Delta M_2$, where $M_1 \Delta M_2$ is a symmetric difference of maximal matchings M_1 and M_2. Then, obviously $M_1\big|_S \neq M_2\big|_S$.

CASE 2. Suppose $e^{(1)} \in M_1 \cap M_2$. Let G' be the graph obtained from $G = CP_{c,4,n-4c}$ by deleting the vertices $u_4^{(1)}$ and v_1. Obviously, G' has 1 connected component which is a path on 3 vertices (i.e. path $P_3^{(1)} = u_1^{(1)}u_2^{(1)}u_3^{(1)}$), $c - 1$ connected components which are cycle on 4 vertices (i.e. cycles $C_4^{(i)} = u_1^{(i)}u_2^{(i)}u_3^{(i)}u_4^{(i)}$) and possibly one connected component which is a path on $n-4c-1$ vertices (i.e. path $P_{n-4c-1} = v_2 \dots v_{n-4c}$). Obviously, $M_1' = M_1\big|_{G'}$ and $M_2' = M_2\big|_{G'}$ are two different maximal matchings on G'. It is easily verified that S restricted on each connected component of G' which is a cycle is a global forcing set. Furthermore, Corollary 5 implies that S restricted to connected components of G which are paths is also a global forcing set. Therefore, $M_1\big|_S \neq M_2\big|_S$.

CASE 3. Suppose $e^{(1)} \notin M_1 \cup M_2$. Let G' be the graph obtained from $G = CP_{c,4,n-4c}$ by deleting the edge $e^{(1)}$. The graph G' consists of two connected components, one is $G_1' = C_4$ and the other is $G_2' = CP_{c-1,4,n-4c}$. Note that $M_1' = M_1\big|_{G'}$ and $M_2' = M_2\big|_{G'}$ are two different maximal matchings on G'. It is easily verified that $S_1' = S\big|_{G_1'}$ is a global forcing set in G_1', while the inductive hypothesis implies that $S_2' = S\big|_{G_2'}$ is a global forcing set on G_2'. Therefore, $M_1\big|_S \neq M_2\big|_S$. ■

7. Concluding remarks

In this paper we have extended several forcing-related concepts from perfect to maximal matchings. In particular, we considered global forcing sets and we have established several results concerning cardinality of smallest such sets in given classes of graphs.

It seems that there are still several interesting unsolved problems. One such problem is to characterize graphs for which the upper bound of Theorem 7 is achieved with equality. We have showed that this happens for all trees and also for complete bipartite graphs with classes of bipartition of unequal size. It would be also interesting to obtain results for grids and linear polymers. Another possible direction could be to investigate how global forcing sets and numbers behave under various binary operations such as sum, corona, or Cartesian product. In particular, operations of splice, link and gated amalgamation should be studied in order to facilitate deriving recurrences for computing global forcing number for maximal matchings in cactus chains and other unbranched polymers of low connectivity.

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